## MEI Structured Mathematics

## Module Summary Sheets

## Statistics 3 <br> (Version B: reference to new book)

## Topic 1: Continuous random variables

Topic 2: Expectation algebra
Topic 3: Sampling
Topic 4: Interpreting sample data using the Normal distribution
Topic 5: Interpreting sample data using the $t$ distribution
Topic 6: Non-parametric tests of location
Topic 7: The chi squared distribution

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## References:

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## Example 1.2

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References:
Chapter 1
Pages 13-15

## Example 1.5

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Exercise 1B
Q. 7

References:
Chapter 1
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## Example 1.6

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Exercise 1B
Q. 2, 5

Continuous random variables are used to model continuous data.

## Probability density function

The p.d.f., $\mathrm{f}(x)$, of the continuous random variable, $X$, has the following properties:
(i) $\mathrm{f}(x)$ is always $\geq 0$
(ii) the total area under $\mathrm{f}(x)$ is 1 ; i.e. $\int_{\text {all } x} \mathrm{f}(x) \mathrm{d} x=1$
(iii) $\mathrm{P}(a \leq x \leq b)=\int_{a}^{b} \mathrm{f}(x) \mathrm{d} x$

The probability that $X$ takes any specific value is zero. i.e. $\mathrm{P}(X=x)=0$

## Expectation and variance

The mean, or expected value, or expectation , of $X$ is given by
$\mu=\mathrm{E}(X)=\int_{\text {all } x} x \mathrm{f}(x) \mathrm{d} x$
The variance is given by
$\operatorname{Var}(X)=\int_{\text {all all }}(x-\mu)^{2} \mathrm{f}(x) \mathrm{d} x=\int_{x} x^{2} \mathrm{f}(x) \mathrm{d} x-\mu^{2}$
These expressions are similar to those for discrete random variables (See Statistics 1).

## Median and Mode

For continuous random variables, the median is the value that separates the area under the graph of the p.d.f. $\mathrm{f}(x)$ into two equal parts, each of area 0.5 . The mode is the value at which the maximum value of $\mathrm{f}(x)$ occurs.
The median is the solution of $\int_{-\infty}^{m} \mathrm{f}(x) \mathrm{d} x=\frac{1}{2}$ or the solution of $\int_{m}^{\infty} \mathrm{f}(x) \mathrm{d} x=\frac{1}{2}$.

The mode is the value of $x$ which maximises $\mathrm{f}(x)$. Depending on the shape of the graph of the function $\mathrm{f}(x)$, this may or may not be a point at which $\mathrm{f}^{\prime}(x)=0$.
E.g. 1. A continuous random variable has the p.d.f. given by $\mathrm{f}(x)=k x(5-x)$ for $0 \leq x \leq 5$. (N.B. $\mathrm{f}(x) \geq 0)$ Find (i) the value of the constant $k$,
(ii) $\mathrm{P}(1 \leq X \leq 2)$.
(i)
i) $\int_{0}^{5} k x(5-x) \mathrm{d} x=k \int_{0}^{5}\left(5 x-x^{2}\right) \mathrm{d} x=k\left[\frac{5 x^{2}}{2}-\frac{x^{3}}{3}\right]_{0}^{5}$

$$
=k\left(\frac{5^{3}}{2}-\frac{5^{3}}{3}\right)=\frac{125 k}{6}=1
$$

$$
\Rightarrow k=\frac{6}{125}
$$

(ii) $\mathrm{P}(1 \leq X \leq 2)=k \int_{1}^{2}\left(5 x-x^{2}\right) \mathrm{d} x=k\left[\frac{5 x^{2}}{2}-\frac{x^{3}}{3}\right]_{1}^{2}$

$$
\begin{aligned}
& =\frac{6}{125}\left(\left(10-\frac{8}{3}\right)-\left(\frac{5}{2}-\frac{1}{3}\right)\right)=\frac{6}{125}\left(\frac{22}{3}-\frac{13}{6}\right) \\
& =\frac{6}{125} \cdot \frac{31}{6}=\frac{31}{125}
\end{aligned}
$$

E.g. For the example above, find the mean and variance.

$$
\begin{gathered}
\mathrm{E}(X)=\int_{0}^{5} k x^{2}(5-x) \mathrm{d} x=k \int_{0}^{5}\left(5 x^{2}-x^{3}\right) \mathrm{d} x=k\left[\frac{5 x^{3}}{3}-\frac{x^{4}}{4}\right]_{0}^{5} \\
=k\left(\frac{5^{4}}{3}-\frac{5^{4}}{4}\right)=\frac{625 k}{12}=\frac{5}{2}
\end{gathered}
$$

$\operatorname{Var}(X)=\int_{0}^{5} k x^{3}(5-x) \mathrm{d} x-\mu^{2}=k \int_{0}^{5}\left(5 x^{3}-x^{4}\right) \mathrm{d} x-\mu^{2}$
$=k\left[\frac{5 x^{4}}{4}-\frac{x^{5}}{5}\right]_{0}^{5}-\left(\frac{5}{2}\right)^{2}=k\left(\frac{5^{5}}{4}-\frac{5^{5}}{5}\right)-\frac{25}{4}$
$=\frac{5^{5} k}{20}-\frac{25}{4}=\frac{25 \times 6}{20}-\frac{125}{20}=\frac{25}{20}=\frac{5}{4}$
E.g. For the example above, the p.d.f. is a parabola.
Therefore the median is the midpoint, because of symmetry, and the mode is the maximum value which is also the mid-point.
So in this case median $=$ mode $=2.5$
E.g. Show that $\mathrm{f}(x)=\frac{1}{8} x$ for $0 \leq x \leq 4, \mathrm{f}(x)=0$ elsewhere, satisfies the conditions for a p.d.f. and find the median and mode. (N.B. $\mathrm{f}(x) \geq 0$ )
$\int_{0}^{4} \frac{1}{8} x \mathrm{~d} x=\left[\frac{x^{2}}{16}\right]_{0}^{4}=1$
Median is $m$ where $\int_{0}^{m} \frac{1}{8} x \mathrm{~d} x=0.5 \Rightarrow\left[\frac{x^{2}}{16}\right]_{0}^{m}=0.5$

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$\Rightarrow \frac{m^{2}}{16}=0.5 \Rightarrow m^{2}=8 \Rightarrow m \approx 2.83$ (3 s.f.)
There is no local maximum for this p.d.f. and so the mode is the largest value of $x$, at the upper limit, $x=4$.

References:
Chapter 1
Pages 18-21

Example 1.7
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Exercise 1B
Q. 8

References:
Chapter 1
Pages 27-30

Example 1.8
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Exercise 1C
Q. 2, 4

References:
Chapter 1
Pages 34-42

Example 1.10
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Exercise 1D
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The Rectangular distribution is also called the continuous uniform distribution.
If $X \sim U(a, b)$ then $\mathrm{f}(x)=\frac{1}{b-a}$ where $a<x<b$ and $\mathrm{f}(x)=0$ for all other values of $x$.
$\mathrm{E}(X)=\frac{a+b}{2}$ and $\operatorname{Var}(X)=\frac{(b-a)^{2}}{12}$
$\mathrm{P}(c<X<d)=\frac{d-c}{(b-a)}$ provided $a<c<d<b$

## Functions of $X$

If the random variable $X$ has p.d.f. $\mathrm{f}(x)$
and $\mathrm{g}[X]$ is a function of $X$ then
$\mathrm{E}(\mathrm{g}[X])=\int_{\text {all } x} \mathrm{~g}[X] \mathrm{f}(x) \mathrm{d} x$
$\operatorname{Var}(\mathrm{g}[X])=\int_{\text {all } x} \mathrm{~g}([x])^{2} \mathrm{f}(x) \mathrm{d} x-\{\mathrm{E}(\mathrm{g}[X])\}^{2}$

General results
$\mathrm{E}(c)=c, \quad \operatorname{Var}[c]=0$
$\mathrm{E}(a X)=a \mathrm{E}(X) \quad \operatorname{Var}(a X)=a^{2} \operatorname{Var}(X)$
$\mathrm{E}(a X+b)=a \mathrm{E}(X)+b \quad \operatorname{Var}(a X+b)=a^{2} \operatorname{Var}(X)$
$\mathrm{E}[\mathrm{g}(X)+\mathrm{h}(X)]=\mathrm{E}[\mathrm{g}(X)]+\mathrm{E}[\mathrm{h}(X)]$

The cumulative distribution function (c.d.f.) of the random variable $X$ is defined by
$\mathrm{F}(x)=\mathrm{P}(X \leq x)=\int_{-\infty}^{x} \mathrm{f}(t) \mathrm{d} t$
$\mathrm{F}(x)$ increases from 0 to 1 as $x$ ranges from $-\infty$ to $+\infty$.
Conversely, $\mathrm{f}(x)=\frac{\mathrm{d}}{\mathrm{d} x} \mathrm{~F}(x)=\mathrm{F}^{\prime}(x)$.
The median of $X, m$, is a root of $\mathrm{F}(m)=\frac{1}{2}$.

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E.g. 1. A Uniform distribution over $[0,3]$ has

$$
\mathrm{f}(x)=\frac{1}{3}
$$

$\mathrm{E}(X)=\frac{0+3}{2}=1.5, \quad \mathrm{E}(X)=\frac{(3-0)^{2}}{12}=0.75$
$\mathrm{P}(1<X<2)=\frac{2-1}{3-0}=\frac{1}{3}$
E.g. For the example above, find
(i) $\mathrm{E}(2 X+5)$,
(ii) $\operatorname{Var}(2 X+5)$
(iii) $\mathrm{E}(2 X-5)$,
(iv) $\operatorname{Var}(2 X-5)$
(i) $\mathrm{E}(2 X+5)=2 \mathrm{E}(X)+5=2 \times 1.5+5=8$
(ii) $\operatorname{Var}(2 X+5)=4 \operatorname{Var}(X)=4 \times 0.75=3$
(iii) $\mathrm{E}(2 X-5)=2 \mathrm{E}(X)-5=2 \times 1.5-5=-2$
(iv) $\operatorname{Var}(2 X-5)=4 \operatorname{Var}(X)=4 \times 0.75=3$
E.g. The p.d.f. of a functon is given by
$\mathrm{f}(x)=k x$ for $0 \leq x \leq 2, \mathrm{f}(x)=0$ elsewhere.
Find
(i) the value of $k$,
(ii) $\mathrm{E}(X)$,
(iii) $\operatorname{Var}(X)$,
(iv) the c.d.f., $\mathrm{F}(x)$,
(v) the median.
(i) $\int_{0}^{2} k x \mathrm{~d} x=\left[\frac{k x^{2}}{2}\right]_{0}^{2}=2 k=1 \Rightarrow k=\frac{1}{2}$
(ii) $\mathrm{E}(X)=\int_{0}^{2} x \cdot k x \mathrm{~d} x=\left[\frac{k x^{3}}{3}\right]_{0}^{2}=\frac{8 k}{3}=\frac{4}{3}$
(iii) $\operatorname{Var}(X)=\int_{0}^{2} x^{2} \cdot k x \mathrm{~d} x-\mathrm{E}(X)^{2}=\left[\frac{k x^{4}}{4}\right]_{0}^{2}-\frac{16}{9}$

$$
=2-\frac{16}{9}=\frac{2}{9}
$$

(iv) the c.d.f., $\mathrm{F}(x)=\int_{0}^{x} k t \mathrm{~d} t=\left[\frac{k t^{2}}{2}\right]_{0}^{x}=\frac{1}{4} x^{2}$
(v) the median, $m$, is a root of the equation
$\mathrm{F}(m)=\frac{1}{2}$.
$\Rightarrow \frac{1}{4} m^{2}=\frac{1}{2} \Rightarrow m^{2}=2 \Rightarrow m=\sqrt{2} \approx 1.41$
(given that $m \geq 0$ )
Alternatively, $m$ satisfies $\int_{0}^{m} k x d x=\frac{1}{2}$

$$
\Rightarrow\left[\frac{x^{2}}{4}\right]_{0}^{m}=\frac{1}{2} \Rightarrow m^{2}=2
$$

References:
Chapter 2
Pages 53-54

Example 2.1
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References:
Chapter 2
Pages 55-61

Exercise 2A
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References:
Chapter 2
Pages 61-63
Pages 65-68
Example 2.6
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References: Chapter 2
Pages 75, 76

Example 2.11 Page 76

Expectation of a function of $X, \mathrm{E}(\mathrm{g}[X])$
$\mathrm{g}[X]$ is a function of the random variable $X$. If $X$ is discrete then $\mathrm{E}(\mathrm{g}[X])$ is given by

$$
\mathrm{E}(\mathrm{~g}[X])=\sum \mathrm{g}\left[x_{i}\right] \times \mathrm{P}\left(X=x_{i}\right)
$$

If $X$ is continuous then $\mathrm{E}[\mathrm{g}(X)]$ is given by

$$
\mathrm{E}(\mathrm{~g}[X])=\int_{-\infty}^{\infty} \mathrm{g}[x] \mathrm{f}(x) \mathrm{d} x
$$

## Algebraic results

$\mathrm{E}(a X+b Y)=a \mathrm{E}(X)+b \mathrm{E}(Y)$
$\mathrm{E}(\mathrm{f}[X]+\mathrm{g}[X])=\mathrm{E}(\mathrm{f}[X])+\mathrm{E}(\mathrm{g}[X])$
If, in addition, the random variables $X$ and $Y$ are independent then
$\operatorname{Var}(a X \pm b Y)=a^{2} \operatorname{Var}(X)+b^{2} \operatorname{Var}(Y)$
In particular, if the random variables $X_{1}, X_{2}$ are independent,

$$
\begin{aligned}
\mathrm{E}\left(X_{1}+X_{2}\right) & =\mathrm{E}\left(X_{1}\right)+\mathrm{E}\left(X_{2}\right) \\
\operatorname{Var}\left(X_{1}+X_{2}\right) & =\operatorname{Var}\left(X_{1}\right)+\operatorname{Var}\left(X_{2}\right) \\
\mathrm{E}\left(X_{1}-X_{2}\right) & =\mathrm{E}\left(X_{1}\right)-\mathrm{E}\left(X_{2}\right) \\
\operatorname{Var} \quad\left(X_{1}-X_{2}\right) & =\operatorname{Var}\left(X_{1}\right)+\operatorname{Var}\left(X_{2}\right)
\end{aligned}
$$

In general, if the random variables $X_{1}, X_{2}, \ldots$ are independent,
$\mathrm{E}\left(a_{1} X_{1} \pm a_{2} X_{2} \pm \ldots \ldots \pm a_{n} X_{n}\right)$

$$
=a_{1} \mathrm{E}\left(X_{1}\right) \pm a_{2} \mathrm{E}\left(X_{2}\right) \pm \ldots . \ldots a_{n} E\left(X_{n}\right)
$$

$\operatorname{Var}\left(a_{1} X_{1} \pm a_{2} X_{2} \pm \ldots \ldots \pm a_{n} X_{n}\right)$

$$
=a_{1}{ }^{2} \operatorname{Var}\left(X_{1}\right)+a_{2}{ }^{2} \operatorname{Var}\left(X_{2}\right)+\ldots . .+a_{n}{ }^{2} \operatorname{Var}\left(X_{n}\right)
$$

## Sums and differences of Normal distributions

If the independent random variables $X_{1}$ and $X_{2}$ are Normally distributed, then the distributions of $a X_{1}+b X_{2}$ and $a X_{1}-b X_{2}$ are also Normally distributed.

## Distribution of sample mean

If the population mean is $\mu$ and the variance is $\sigma^{2}$ and a random sample of $n$ items, $x_{1}, x_{2}, \ldots, x_{n}$, from the population is taken, then

$$
\bar{x}=\frac{x_{1}+x_{2}+\ldots . .+x_{n}}{n}
$$

$\bar{x}$ is a value of the random variable $\bar{X}=\frac{X_{1}+X_{2}+\ldots . .+X_{n}}{n}$

$$
\begin{aligned}
\Rightarrow \mathrm{E}(\bar{X}) & =\frac{1}{n} \mathrm{E}\left(X_{1}\right)+\frac{1}{n} \mathrm{E}\left(X_{2}\right)+\ldots . .+\frac{1}{n} \mathrm{E}\left(X_{n}\right) \\
& =\frac{1}{n}, \mu+\frac{1}{n} \cdot \mu+\ldots .+\frac{1}{n} . \mu=\mu
\end{aligned}
$$

Given also that $X_{1}, X_{2}, \ldots . X_{n}$ are independent,

$$
\begin{aligned}
\operatorname{Var}(\bar{X}) & =\left(\frac{1}{n}\right)^{2} \operatorname{Var}\left(X_{1}+X_{2}+\ldots .+X_{n}\right) \\
& =\frac{1}{n^{2}}\left(\operatorname{Var}\left(X_{1}\right)+\operatorname{Var}\left(X_{2}\right)+\ldots+\operatorname{Var}\left(X_{n}\right)\right) \\
& =\frac{1}{n^{2}} \cdot\left(\sigma^{2}+\sigma^{2}+\ldots+\sigma^{2}\right)=\frac{1}{n^{2}} \times n \sigma^{2}=\frac{1}{n} \sigma^{2}
\end{aligned}
$$

E.g. The random variable $X$ has a probability distribution as shown.

| $x$ | 135 |  |  |
| :---: | :---: | :---: | :---: |
| $\mathrm{P}(X=x)$ | 0.50. | 30. | 2 |

Find $\mathrm{E}(2 X+3)$ by two methods.
Method 1.
By the work of Statistics 1;
$\mathrm{E}(2 X+3)=2 \mathrm{E}(X)+3$
And $\mathrm{E}(X)=1 \times 0.5+3 \times 0.3+5 \times 0.2=2.4$
$\Rightarrow \mathrm{E}(2 X+3)=2 \times 2.4+3=7.8$

## Method 2.

$\mathrm{g}[X]=2 X+3$ gives the table as shown.

| $x 1$ |  | 3 | 5 |
| :---: | :---: | :---: | :---: |
| $\mathrm{~g}[x] 5$ |  | 9 | 13 |
| $\mathrm{P}(X=x)$ | 0. | 5 | 0.3 |

(When $x=1, \mathrm{~g}(x)=2 \times 1+3=5$, etc.)
$\Rightarrow \mathrm{E}(2 X+3)=5 \times 0.5+9 \times 0.3+13 \times 0.2$

$$
=7.8
$$

E.g The random variable $X$ is Normally distributed.
Then $\mathrm{E}(2 X)=2 \mathrm{E}(X), \operatorname{Var}(2 X)=4 \operatorname{Var}(X)$
The independent random variables $X_{1}$ and $X_{2}$ are drawn from the same random variable $X$.
Then $\mathrm{E}\left(X_{1}+X_{2}\right)=\mathrm{E}\left(X_{1}\right)+\mathrm{E}\left(X_{2}\right)=2 \mathrm{E}(X)$
and
$\operatorname{Var}\left(X_{1}+X_{2}\right)=\operatorname{Var}\left(X_{1}\right)+\operatorname{Var}\left(X_{2}\right)=2 \operatorname{Var}(X)$
E.g. The distribution of masses in kilograms of male students at a college is $\mathrm{N}(75,5)$ and of females is $\mathrm{N}(70,8)$.
A male and a female enter a lift together. Assuming the two students to be a random choice from the populations, find the probability that
(i) the mass of the two students is less than 155 kg ,
(ii) the female is heavier than the male.
(i) $X_{m} \sim \mathrm{~N}(75,5), X_{f} \sim \mathrm{~N}(70,8)$
$\Rightarrow X_{m}+\mathbf{N}(f \sim 145,13)$
$\Rightarrow \mathrm{P}\left(X_{m}+X 55,5<\quad\right)=\Phi\left(\frac{155-145}{\sqrt{13}}\right)$
$=\Phi(2.7735)=0.9972$
(ii) $X_{m}-X_{f} \sim \mathrm{~N}(5,13)$
$\Rightarrow \mathrm{P}\left(X_{m}-\boldsymbol{\Omega}_{f}<\right)=\Phi\left(\frac{-5}{\sqrt{13}}\right)$
$=1-\Phi(1.3868)=1-0.9172=0.0828$

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## References:

Chapter 3
Pages 80,81

## References: <br> Chapter 3 <br> Page 81

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Pages 81-84


## Terms

The population or parent population is the set of all data in which you are interested.
A sample is a set of data drawn from the parent population in order to investigate the population. Usually the population is too large (or even infinite) or too difficult (or costly) to collect or to investigate and so we investigate using a smaller set which needs to be as representative as possible.
A representation of the items available to be sampled is called the sampling frame.
The proportion of available items that are actually sampled is called the sampling fraction.

## Notation

A statistical value that describes the set (for instance mean or variance) is called a (sample) statistic if it is found from the sample and a (population) parameter if it is found from the population.
For a population, greek letters are used.

$$
\text { e.g. } \text { mean }=\mu, \text { variance }=\sigma^{2}
$$

For a sample, roman letters are used.
e.g. mean $=\bar{x}$, variance $=s^{2}$

When the parameters of a sample are used for the population then they are called estimates. Upper case letters, $X, Y$, etc are used for random variables and lower case letters, $x, y$, etc for particular values.

## Sampling

The reason for taking a sample is:
(i) In order to estimate the values of the parameters of the population when to find the actual values is too costly, time consuming or impossible.
(ii) To conduct a hypothesis test.

Questions to ask.
Are the data relevant?
Are the data biased?
Does the method of collecting the data create bias or distortion?
Is the sample large enough?
Is the sampling procedure appropriate?

## Sampling techniques

Simple random sampling. Every possible sample of given size is equally likely to be selected.
Stratified sampling. When the population is split into distinct strata, then separate samples (often simple random, and sometimes in proportion) are chosen from each strata.
Cluster sampling. The population is split into groups or clusters. A simple random sample (often only of size one) of clusters is chosen. The chosen clusters are then surveyed - often by studying all the elements in them, or by taking random samples within them. Systematic sampling. This is a method of choosing a sample from a sampling frame by a systematic method - e.g. taking every 3 rd member of a list. Quota sampling. A sample is taken that fulfils some criteria of quotas of certain sub-groups within the population.
E.g. If a sample of size 10 is chosen from a population of size 200 then the sampling fraction is

$$
\frac{10}{200}=\frac{1}{20}
$$

To have the same sampling fraction for a population of size 5000 requires a sample of size

$$
5000 \times \frac{1}{20}=250
$$

## Simple random sampling

E.g. The choice of a simple random sample may be made using random numbers from the Students Handbook (page 52).
For numbers between 1 and 100 take numbers in pairs (taking 00 to be 100) or add 1 to each number generated.
For numbers between 1 and 800 take digits in threes and reject any number obtained that is greater than 800 ; also reject 000 .

Random numbers can also be generated from most calculators. These are generally a three digit decimal number between 0 and 1. If you want numbers between 1 and 800 , multiply by 800 , ignore the decimal part and add 1 to each.

## Stratified Sampling

E.g. Split a student population into male and female.
E.g. Split a work force into the "managers" and "machine operators".

## Cluster Sampling

E.g. To investigate all students studying Mathematics in Colleges of Further Education, take as a sample all students (or a simple random sample of students) in a random selection of colleges. The colleges are the clusters.

## Systematic Sampling

E.g. Select a sample of size 10 from 200. $200 / 10=20$. Take a number randomly in the range $1-20$ then add 20 to each number.
E.g. 13 , so take the $13^{\text {th }}, 33^{\text {rd }}, 53^{\text {rd }}$, etc.

## Quota Sampling

E.g. A poll is to be carried out on the population of the UK on drinking habits.
A number of people are asked to conduct the poll. Each one has to interview 20 females aged under 20, 20 who are in the age bracket $20-40$ and 20 aged over 40 . They have to make the same selection for men.

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References:
Chapter 4
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## References:

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## Example 4.1 <br> Page 98

Exercise 4A
Q. 1, 3

## References:

Chapter 4 Page 95

## References:

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Exercise 4A
Q. 4

The Central Limit Theorem
If $X_{1}, X_{2}, \ldots X_{n}$ are independent and identically distributed and come from a (usually unknown) distribution with mean $\mu$ and variance $\sigma^{2}$ then approximately:

$$
\bar{X} \sim \mathrm{~N}\left(\mu, \frac{\sigma^{2}}{n}\right) \text { add } T=X_{1}+X_{2}+\ldots+X_{n} \sim \mathrm{~N}\left(n \mu, n \sigma^{2}\right)
$$

The approximation to a Normal distribution improves as the sample size, $n$, increases.
Usually a sample size of at least 25 is adequate.
Such samples are called "large".
$\frac{\sigma}{\sqrt{n}}$ is the standard deviation of $\bar{X}$. It is sometimes
called the standard error of the mean.

## Confidence intervals

A 95\% confidence interval for the mean of the population is an interval calculated from a random sample. A different sample would give a different interval. If lots of samples were taken from the population, $95 \%$ of the intervals calculated would
contain the fixed, but unknown, mean of the population. If we have a sample of size $n$ from a Normal distribution whose variance is known, the symmetrical $100(1-\alpha) \%$ confidence limits for the mean of the population are given by

$$
\bar{x} \pm k \frac{\sigma}{\sqrt{n}} \text { where } \mathrm{P}(-k<Z<k)=1-\alpha
$$

Values of $k$ will depend on the confidence level
Confidence level $k$

| $90 \%$ | 1.645 |
| :--- | :--- |
| $95 \%$ | 1.96 |
| $99 \%$ | 2.576 |

The confidence interval is from $\bar{x}-k \frac{\sigma}{\sqrt{n}}$ to $\bar{x}+k \frac{\sigma}{\sqrt{n}}$

## Known and estimated standard deviation

In the work above the value of the standard deviation of the population is needed. If the standard deviation is not known then it has to be estimated from the sample. In this case it is acceptable to use the s.d. from the sample as an estimate and assume that the sampling distribution of means is approximately Normal provided that the sample size is at sufficiently large (a sample size of about 50 is often adequate.)

## Size of sample.

In all investigations where a sample is required, the larger the sample the more confidence you have in the results, but this needs to be balanced against the cost and the time of collecting.
The width of the confidence interval is inversely proportional to the square root of $n$, so we need to quadruple the size of the sample in order to halve the width.

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E.g. Assume that the marks of all candidates in Statistics 3 in a particular college are Normally distributed with variance 100 .
If samples of size 20 are taken then the Standard Error of the mean is

$$
\frac{10}{\sqrt{20}}=2.236
$$

If samples of size 200 are taken then the Standard Error of the mean is

$$
\frac{10}{\sqrt{200}}=0.7071
$$

E.g. In the college above, the mean mark of one sample of 200 candidates is 58 , then the $95 \%$ confidence limits are

$$
\begin{aligned}
& 58 \pm \frac{1.96 \times 10}{\sqrt{200}} \\
& \text { i.e. } 56.6 \text { and } 59.4
\end{aligned}
$$

E.g. If $\bar{X} \sim \mathrm{~N}(\mu, 25)$ and a random sample of size 16 from it has mean 40 , then the $99 \%$ confidence interval for $\mu$ is
$40 \pm 2.576 \times \frac{\sqrt{25}}{\sqrt{16}}=40 \pm 3.22$
i.e. 36.88 to 43.22 .
E.g. If $\bar{X} \sim \mathrm{~N}(\mu, 25)$ and a random sample of size 100 from it has mean 40 , then the $99 \%$ confidence interval for $\mu$ is
$40 \pm 2.576 \times \frac{\sqrt{25}}{\sqrt{100}}=40 \pm 1.29$
i.e. 38.71 to 41.29 .
E.g. In the college above, the investigator would like to be able to estimate the mean mark within 2 marks with $99 \%$ confidence. What size sample should she take?

The $99 \%$ confidence interval for the mean
is given by $\bar{x}-2.58 \frac{\sigma}{\sqrt{n}}$ to $\bar{x}+2.58 \frac{\sigma}{\sqrt{n}}$,
where $\delta=0$.
$\Rightarrow \frac{25.8}{\sqrt{n}} \leq 2$
$\Rightarrow n \geq\left(\frac{25.8}{2}\right)^{2} \Rightarrow n \geq 166.4$
$\Rightarrow$ at least 167

References:
Chapter 5
Page 106-107

## References:

Chapter 5
Pages 107-108

References:
Chapter 5
Pages 108-110

Exercise 5A
Q. 2, 4

References:
Chapter 5
Pages 110-112
Example 5.1
Page 111

Exercise 5A
Q. 3, 6

## Interpreting the sample data

For situations where the population mean, $\mu$, and variance, $\sigma^{2}$ are both unknown, sample data may be interpreted using the $t$ distribution provided the distribution from which they are drawn is Normal.
The $t$ distribution is only needed if the sample size is small; for large samples the Normal distribution may be used because of the Central Limit Theorem.

## Degrees of freedom

The number of free variables in the situation is called the degrees of freedom.
As the sample size increases, the estimate $s$ will tend to approach the true value of $\sigma$ and the relevant $t$ distribution approaches the standard Normal distribution. The $t$ distribution which we use is specified by its "degrees of freedom", $v$. In this situation, $v=n-1$.

## Confidence intervals

Just as in the case above, if we have a random sample of size $n$ from a Normal population where we do not know the value of $\sigma$, the symmetric $100(1-\alpha) \%$ confidence limits for the mean of the population are given by

$$
\begin{gathered}
\bar{x} \pm \frac{k s}{\sqrt{n}} \\
s^{2}=\frac{\sum(x-\bar{x})^{2}}{n-1} \text { is an "unbiased" estimate of the }
\end{gathered}
$$

(unknown) population variance from the sample data.
If you were to take lots of samples and calculate $s^{2}$ for each sample the values would centre on the correct value, $\sigma^{2}$.
$k$ is found from $t$ tables with $n-1$ degrees of freedom.
Then $\mathrm{P}\left(-k<t_{n-1}<k\right)=1-\alpha$
If the sample size is large the central limit theorem means that the confidence limits are

$$
\bar{x} \pm \frac{k s}{\sqrt{n}}
$$

where $k$ is found from the Normal table and $s^{2}$ is used in place of the population variance if the latter is not known.

## Hypothesis test using the $\boldsymbol{t}$ distribution

As before, $\mathrm{H}_{0}$ is $\mu=\mu_{0}$ and $\mathrm{H}_{1}$ may be one tailed, $\mu<\mu_{0}$ or $\mu>\mu_{0}$, or two tailed: $\mu \neq \mu_{0}$.

The test statistic is

$$
T=\frac{\bar{X}-\mu_{0}}{s / \sqrt{n}}
$$

with $s$ as above and the $t$ tables with $n-1$ degrees of freedom are used. This $t$ test may only be used for random samples from a Normal population.
E.g. A machine packs bags of sugar of nominal mass 1000 g . A sample of size 50 has mean mass 999.3 g and s.d. 2.3 g . Is there evidence at the $5 \%$ level that bags are being underfilled?
$\mathrm{H}_{0}: \mu=1000$
$\mathrm{H}_{1}: \mu<1000$
The value of the test statistic is

$$
z=\frac{\bar{x}-\mu_{0}}{s / \sqrt{n}}=\frac{999.3-1000}{2.3 / \sqrt{50}}=-2.152
$$

Because the sample size is large enough, we may use Normal Tables which gives the 5\% critical value to be -1.645 . We therefore reject $\mathrm{H}_{0}$ - there is evidence to suggest that the bags are being underfilled.
E.g. 6 students are asked to estimate a time interval of 1 minute. Their estimates, in seconds, are:
55.2, 67.3, 53.9, 59.2, 65.0, 63.7

Assuming that these form a random sample from a Normal population, the $90 \%$ confidence limits for the mean are:
$\bar{x} \pm \frac{k s}{\sqrt{n}}=60.72 \pm 2.015 \times \frac{5.47}{\sqrt{6}}=56.2$ to 65.2
where $\bar{x}$ and $s$ (the sample s.d. with
divisor $n-1$ ) from a calculator are 60.72
and 5.47 and $k$ from $t_{5}$ tables is 2.015 .
E.g. The mean mass of adult females of a certain species of bird is 16.2 g .
An ornithologist finds four similar looking birds of masses $15.9,17.8,18.2,19.7 \mathrm{~g}$. Is there evidence at the $10 \%$ level that they may be from a different species?
$\mathrm{H}_{0}: \mu=16.2$
$\mathrm{H}_{1}: \mu \neq 16.2$
Assuming a random sample from a Normal population the value of the test statistic is

$$
t=\frac{\bar{x}-\mu_{0}}{s / \sqrt{n}}=\frac{17.9-16.2}{1.564 / \sqrt{4}}=2.174
$$

From $t_{3}$ tables, the $10 \%$ critical value is 2.353 so we accept $\mathrm{H}_{0}$ at the $10 \%$ level.

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References:
Chapter 5
Page 121-124

## References: <br> Chapter 5 <br> Page 126

References:
Chapter 5
Page 127

Example 5.2
Page 127

References:
Chapter 5
Page 128

Exercise 5B
Q. 2, 5

## The paired sample $\boldsymbol{t}$ test

The test in this case is on a hypothesis related to the difference in 2 characteristics of each member of a population. Each element therefore produces a pair of values.
If the random variable, $D$, is the difference (assumed to be Normal) and $S^{2}$ is the unbiased estimator of its variance based on a sample of size $n$ then the test statistic, $T$, is given by

$$
T=\frac{\bar{D}}{S / \sqrt{n}}
$$

which has a $t$ distribution with $n-1$ degrees of freedom.

## Important assumptions

1. The sample is random.
2. The difference is Normally distributed.

Testing for a non-zero value, $k$, of the difference of two means with paired samples
The test statistic in this case is

$$
T=\frac{\bar{D}-k}{S / \sqrt{n}}
$$

where $\mathrm{H}_{0}: \mu_{D}=k \quad \mathrm{H}_{1}: \mu_{D} \neq k$ (or $\mu_{D}<k$ or $\mu_{D}>k$ ) Once again, we assume $D$ to be Normally distributed $-T$ then has a $t$ distribution with $n-1$ degrees of freedom.

## Confidence intervals for the difference of two

 means from a paired sampleThe symmetrical confidence interval for the mean difference between the characteristics is

$$
\bar{d} \pm k \times \frac{s}{\sqrt{n}}
$$

where the value of $k$ is found from the $t$ distribution tables (Students handbook, page 45) for the appropriate level of significance and the number of degrees of freedom.

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E.g. 6 cars are used to assess the effect of a new fuel additive designed to reduce fuel consumption.
Under the same driving conditions, the total consumption figures, in litres per 100 km , are as follows.
( $x=$ without additive, $y=$ with additive)

| Car | A | B | C | D | E | F |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $x$ | 9.2 | 8.3 | 5.8 | 5.9 | 4.6 | 7.7 |
| $y$ | 8.3 | 7.5 | 5.1 | 5.4 | 4.8 | 7.8 |
| $d$ | -0.9 | -0.8 | -0.7 | -0.5 | 0.2 | 0.1 |

Is there evidence at the $5 \%$ level that the additive is effective?
$\mathrm{H}_{0}: \mu_{D}=0$ (no effect)
$\mathrm{H}_{1}: \mu_{D}<0$ (reduced fuel consumption)
For these data $\sum d=-2.6 \Rightarrow \bar{d}=-0.433$
$\sum d^{2}=2.24 \Rightarrow s=0.472$
$\Rightarrow$ Test statistic, $t=\frac{\bar{d}}{s / \sqrt{n}}=\frac{-0.433}{0.472 / \sqrt{6}}=-2.247$
For $t_{5}, 5 \%, 1$-tail test the critical value $=-2.015$. Since $-2.247<-2.015$, we reject $\mathrm{H}_{0}$ in favour of $\mathrm{H}_{1}$; there is evidence to suggest that consumption is less with the additive.
E.g. For the same data, is there evidence at the $5 \%$ level of significance that the additive reduces the fuel consumption by at least 0.25 litres per 100 km ?
$\mathrm{H}_{0}: \mu_{D}=-0.25$
$\mathrm{H}_{1}: \mu_{D}<-0.25$
$\Rightarrow$ Test statistic, $t=\frac{\bar{d}-(-0.25)}{s / \sqrt{n}}=-0.950$
As before, the critical value is -2.015 and since $-0.950>-2.015$ we accept $\mathrm{H}_{0}$.
E.g. For the same data, the $90 \%$ confidence limits for the mean change in fuel consumption with the additive are
$\bar{d} \pm 2.015 \times \frac{s}{\sqrt{n}}=-0.433 \pm 2.015 \times \frac{0.472}{\sqrt{6}}$
i.e. -0.821 and -0.044

References:
Chapter 6
Pages 135-136

References:
Chapter 6
Pages 137-143

## Exercise 6A

Q. 1, 4

Example 6.1
Page 142

## The sign test

This is the simplest of tests. It seeks to discover a trend of those in a sample who agree or disagree or who say yes or no.
The hypothesis is usually that equal numbers fall into the two categories.
On that basis, the distribution will be $\mathrm{B}(n, 0.5)$ and a test can be carried out using the cumulative binomial tables (as in the module Statistics 1).

## The Wilcoxon single sample test

The survey may be more complicated than simply agree or disagree. There may be categories such as "strongly agree" or a value may be given, say 1-10.
This Wilcoxon test takes the "difference" of the value of the response from the median value and then ranks their absolute values.
If the median is the "usual" response, then the distribution will be symmetric about this value. (So one above and one below will receive equal rankings.) Also the sum of rankings above the median will be equal to the sum of rankings below.
The assumptions are as follows:
The random variable is symmetrically distributed about the median, $M$.
The data form a random sample of size $n$. If any value is equal to the median, then remove it and reduce the value of $n$.
$\mathrm{H}_{0}$ : The population median equals $M$.
$\mathrm{H}_{1}$ : can be one tailed (the population median $>M$ or $<M$ ) or two tailed (the population median $\neq M$ ).
Calculation is as follows:
Calculate the absolute differences between each sample value and the hypothesised median, deleting those that are equal to this median.
Rank these values from 1 to $n$ giving the lowest rank to the smallest absolute difference. If any are equal then give each the average of the ranking positions that they occupy together. Calculate the sum $W_{+}$of the ranks of the values above $m$, and the sum, $W_{-}$of the ranks of the sample values below $M$.
Check that $W_{+}+W_{-}$equals the sum of integers $1, \ldots, n$ which is

$$
\frac{1}{2} n(n+1)
$$

For the two tailed hypothesis, the test statistic, $W$ is the smaller of $W_{+}$and $W_{-}$. For the one tailed test $W$ is one or the other as appropriate.
Critical values are given in the table in the Students' Handbook
(Page 52) for the smallest value that one of the sums could be before rejecting the null hypothesis that there is no strong opinion either way.

## References:

Chapter 6
Pages 149-150

## Example 6.3 <br> Page 151

## Exercise 6B

Q. 2, 4

## The Wilcoxon paired sample test.

This concerns the testing of two conditions, giving a pair of values for each member of a population.
Find the differences between the values in the two conditions. Use the Wilcoxon single sample test with the hypothesis that these differences have the suggested median (often zero). between the values of the random variables in the two conditions are symmetrically distributed about the differences in medians.

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E.g. 10 people taste butter and a butter substitute. They are asked to identify which is which. If 8 are correct, is there evidence at the $5 \%$ level that people can tell butter from the substitute?
$\mathrm{H}_{0}$ : There is no evidence that people can tell the difference
$\mathrm{H}_{1}$ : There is evidence that people can tell the difference.
From B(10, 0.5 ) tables the 1-tail $5 \%$ critical region is $\{9,10\}$; since 8 is not in this region we can accept $\mathrm{H}_{0}$; there is no evidence at the $5 \%$ level that people can tell butter from the substitute.
E.g. A newspaper article claimed that teachers work an average of 38 hours a week. 10 teachers were asked how many hours they worked in one week and their responses were as follows.
$\begin{array}{llllllllll}34 & 36 & 38 & 39 & 42 & 45 & 47 & 52 & 53 & 55 .\end{array}$ Test, at the $5 \%$ level of significance, the newspaper article's claim, using the median as "average" against the alternative hypothesis that teachers work longer hours than 38 hours a week.
$\mathrm{H}_{0}: M=38 \quad \mathrm{H}_{1}: M>38$
Values of $M-38$ :
$\begin{array}{lllllllll}-4 & -2 & 1 & 4 & 7 & 9 & 14 & 15 & 17\end{array}$
Ranked absolute differences are:
$\begin{array}{lllllllll}3.5 & 2 & 1 & 3.5 & 5 & 6 & 7 & 8 & 9\end{array}$
$\mathrm{W}_{-}=5.5 \quad \mathrm{~W}_{+}=39.5$
Check: $5.5+39.5=\frac{1}{2} \times 9 \times 10$
For $n=9$, the $5 \% 1$-tail critical value is 8 ; since $5.5<8$ we accept $\mathrm{H}_{1}$ - there is evidence that teachers work longer hours.
E.g. 7 people are asked to have 10 goes at writing the word "Scissors" with their non-dominant hand.

| Person | A B | C | D | E | F | G |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Time for <br> first try | 3.2 | 3.74. | 25. | 2 | 3.83. | 4 | 4.0 |
| Time for <br> 10 th try | 2.6 | 3.44. | 34. | 7 | 3.23. | 6 | 3.2 |
| Difference | 0.6 | $0.3-0.10$. | 5 | $0.6-$ | 0.2 | 0.8 |  |
| Ranked <br> absolute <br> difference | 5.5 | 314 |  | 5.5 | 2 | 7 |  |

Is there evidence, at the $5 \%$ level of significance, that they have improved with practice?
$\mathrm{H}_{0}: M=0, \mathrm{H}_{1}: M>0$ (i.e. 10th try is quicker than 1st.)
$\mathrm{W}_{-}=1+2=3$
For $n=7$, the $5 \% 1$-tail critical value is 3 ; we thus accept $\mathrm{H}_{1}$ - there is evidence that they improve with practice.

References:
Chapter 7
Pages 157-158

Example 7.2
Page 163

Exercise 7A
Q. 4

## References:

Chapter 7
Page 158

## References:

Chapter 7
Page 159

## References: <br> Chapter 7 <br> Page 160

## The Chi squared distribution

The Chi squared ( $\chi^{2}$ ) goodness of fit test may be used to see if a particular model (e.g. binomial, Poisson, Normal) is appropriate for the given data set. If any of the parameters of the model are not specified in the null hypothesis (e.g. the mean of the Poisson model) then they are estimated from the data.
The expected frequencies for each of the possible values are calculated using the model. In the case of models such as the Poisson model, where there are infinitely many possible values, the "tail" values will be put together in one group.
The test is approximate. This approximation is considered "good" if all of the expected frequencies are at least 5 . If this is not the case, classes are pooled/ combined until this condition is satisfied.

The test statistic is $X^{2}=\sum \frac{\left(f_{o}-f_{e}\right)^{2}}{f_{e}}$ where $f_{\mathrm{o}}$ is the observed frequency for a class and $f_{e}$ is the expected frequency, as calculated from the model.

This has a chi squared distribution if the model is correct.

## Significance levels

The significance level of the test is determined by your original hypothesis and not by what you see in the data. Therefore the significance level of a test must be set at the beginning.

## Degrees of freedom

The degree of freedom is the number of cells minus one (for the totals to agree) minus one for each parameter of the model which was estimated from the data.
E.g. Poisson $(\lambda)$ : $v=$ No. of cells -1 if $\lambda$ is given.
$v=$ No. of cells -2 if $\lambda$ is estimated from the data.

## Alternative notation

Rather than $f_{\mathrm{o}}$ and $f_{\mathrm{e}}$ we sometimes use $E_{i}$ and $O_{i}$ to represent the expected and observed frequency in the $i$ th class respectively.
Then $X^{2}=\sum \frac{\left(E_{i}-O_{i}\right)^{2}}{E_{i}}$

Example 1: Five identical dice are thrown 150 times and the number of sixes is recorded.
$\begin{array}{llllllll}\text { No. of sixes } & 0 & 1 & 2 & 3 & 4 & 5\end{array}$
Observed frequency $49 \begin{array}{llllll}62 & 24 & 12 & 3 & 0\end{array}$
Test if the dice are unbiased.
$\mathrm{H}_{0}$ : The outcome can be modelled by the binomial distribution $\mathrm{B}(5,1 / 6)$
$\mathrm{H}_{1}$ : The outcome cannot be modelled by the binomial distribution. $\mathrm{B}(5,1 / 6)$
Expected frequencies are calculated from the binomial distribution:
E.g. for 2 sixes, expected number $=150 \times\binom{ 5}{2}\left(\frac{1}{6}\right)^{2}\left(\frac{5}{6}\right)^{3}=24.1$

Expected values $=60.3,60.3,24.1,4.8,0.5,0.0$
The last 3 values are pooled to give a value greater than 5 .
Sixes $\begin{array}{lllll} & 0 & 1 & 2 & 3+\end{array}$
$\begin{array}{lllll}f_{o} & 49 & 62 & 24 & 15\end{array}$
$\begin{array}{lllll}f_{e} & 60.3 & 60.3 & 24.1 & 5.3\end{array}$
$\begin{array}{lllll}f_{o}-f_{o} & -11.3 & 1.7 & -0.1 & 9.7\end{array}$
$\left(f_{o}-f_{e}\right)^{2} 127.69 \quad 2.89 \quad 0.01 \quad 94.09$
$\begin{array}{lllll}\frac{\left(f_{o}-f_{e}\right)^{2}}{f_{e}} & 2.11 & 0.05 & 0.00 & 17.75\end{array}$
$\Rightarrow$ Test statistic, $X^{2}=19.91 \quad v=4-1($ same total $)=3$
$\Rightarrow$ Reject $\mathrm{H}_{0}$ at $5 \%$ level since critical value $=19.91>7.81$
E.g. The number of cars passing an observer in 64 one-minute intervals is given in the following table.

No. of cars $\begin{array}{llllll}0 & 1 & 2 & 3 & 4\end{array}$
$\begin{array}{llllll}\text { Frequency } & 20 & 19 & 13 & 10 & 2\end{array}$
Test whether these data could have come from a Poisson Distribution.
Sample mean $=1.3$
$\mathrm{H}_{0}$ : The distribution is Poisson with mean 1.3
$\mathrm{H}_{1}$ : The distribution is not Poisson with mean 1.3
The table (with the last two groups combined) is as follows:


[^1]
## References:

Chapter 7
Pages 161-167

Example 7.1
Page 161

Example 7.2
Page 163

References:
Chapter 7
Pages 177-178

## Example 7.4

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Exercise 7B
Q. 2

References:
Chapter 7
Pages 179-181
Example 7.5
Page 179
Exercise 7B
Q. 4

References:
Chapter 7
Page 182

Example 7.6
Page 182

Exercise 7B Q. 5

References:
Chapter 7
Pages 168-169

## Using the test

1. Decide on the model.
2. Set up the hypotheses and decide on the level of significance.
3. Collect the data.
4. Calculate the expected frequencies.
5. Check the size of the classes and pool if necessary.
6. Calculate $X^{2}$.
7. Work out the degrees of freedom.
8. Find the critical value from tables and compare.
9. State your conclusion.

Testing for a given Normal distribution To test for $\mathrm{N}\left(\mu, \sigma^{2}\right)$ carry out the steps as above where the expected frequencies are derived from the standardised Normal distribution tables. To do this you need to decide the number of classes to be used and the width of each. You need to choose these so that each class has a minimum of 5 members.
The degrees of freedom are the number of classes minus 1 , because the sum of the observed and expected frequencies are the same.

## Testing for Normality

First calculate an estimate of the mean and variance of the population given the data in the sample.
Then carry out the steps as above.
The degrees of freedom are two less than above as the mean and variance have been determined by the sample.

## Other distributions

You may be asked to use the Chi squared test to test for the following distributions:

- Uniform
- Binomial
- Poisson
- Normal (with mean and variance known and unknown)
The text also gives examples of exponential distributions.


## The left hand tail

Although the $\chi^{2}$ test is a 1-tailed test (upper tail), very low values for $\chi^{2}$ which would occur in the corresponding lower tail are
sometimes considered as indicating a
suspicious data set when the data fit is "too good to be true."

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E.g. Test the following data at the $5 \%$ level of significance to see if a $N(18,36)$ is a good model.

| Classes | $f_{0}$ | $\mathrm{H}_{0}:$ the data can be modelled by a <br> $\mathrm{N}(18,36)$ distribution <br> $x<5$ |
| :--- | :--- | :--- |
| $\mathrm{H}_{1}$ : the data cannot be modelled <br> by a N $(18,36)$ distribution. |  |  |
| $5 \leq x<10$ | 15 | Expected frequencies are |
| $10 \leq x<15$ | 40 | calculated by finding the Normal <br> $15 \leq x<20$ |
| probability of being in the |  |  |
| particular class and multiplying |  |  |
| $20 \leq x<25$ | 18 | by 120. |
| $x \geq 25$ | 8 | The first two classes need to be <br> pooled as the frequency of the <br> first is less than 5 |

E.g. For the 20-25 class $f_{\mathrm{c}}=\left(\Phi\left(\frac{25-18}{6}\right)-\Phi\left(\frac{20-18}{6}\right)\right) \times 120$

$$
=29.73
$$

| Classes | $f_{\mathrm{o}}$ | $f_{\mathrm{e}}$ | $f_{\mathrm{o}}-f_{\mathrm{e}}$ | $\left(f_{\mathrm{o}}-f_{\mathrm{e}}\right)^{2}$ | $\left(f_{\mathrm{o}}-f_{\mathrm{e}}\right)^{2} / f_{\mathrm{e}}$ |
| :--- | :--- | :--- | :--- | :--- | :--- |
| $x<10$ | 18 | 10.95 | 7.05 | 49.70 | 4.539 |
| $10 \leq x<15$ | 40 | 26.08 | 13.92 | 193.79 | 7.431 |
| $15 \leq x<20$ | 36 | 38.64 | -2.64 | 6.98 | 0.181 |
| $20 \leq x<25$ | 18 | 29.73 | -11.73 | 137.65 | 4.630 |
| $x \geq 25$ | 8 | 14.60 | -6.60 | 43.57 | 2.984 |
| Total | 120 | 120 |  |  | 19.765 |

Test statistic, $X^{2}=19.765$ with $v=5-1($ same total $)=4$ Critical value from tables $=9.488$
Since $19.765>9.488$ we reject $H_{0}$ in favour of $H_{1}$ : there is evidence at the $5 \%$ level of significance that the data comes from a distribution other than $\mathrm{N}(18,36)$.
E.g. The days of birth of 100 footballers are:

Sat Sun Mon Tues Wed Thurs Frid
$\begin{array}{llllllll}f_{\mathrm{o}} & 20 & 17 & 11 & 15 & 10 & 13 & 14\end{array}$
$\begin{array}{llllllll}f_{\mathrm{e}} & 14.3 & 14.3 & 14.3 & 14.3 & 14.3 & 14.3 & 14.3\end{array}$
Test at the $5 \%$ level of significance that these data can be considered to be a uniform distribution.
$\mathrm{H}_{0}$ : Birthdays are uniformly distributed through the week $\mathrm{H}_{1}$ : Birthdays are not uniformly distributed through the week.

For these data: $\sum \frac{\left(f_{\mathrm{o}}-f_{\mathrm{e}}\right)^{2}}{f_{\mathrm{e}}}=4.995$
$v=7-1($ same total $)=6 ;$ critical value $=12.59$
Since $4.995<12.59$ we accept $\mathrm{H}_{0}$ : we conclude that there is evidence to support the hypothesis that the data are uniformly distributed.
E.g. in Example 1 on the previous page, had the observed frequencies been $60,60,24,6$; for left hand tail, using the calculated frequencies of the table gives $X^{2}=0.096$.
For $v=3$ and the same level of significance,
critical value $=0.115$.
Since $0.096<0.115$ this gives rise to suspicion that the data are "too good to be true".


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