MEI

Mathematics in Education and Industry

MEI STRUCTURED MATHEMATICS

NUMERICAL METHODS - NM

Practice Paper NM-A

Additional materials: Answer booklet/paper
Graph paper
List of formulae (MF2)

TIME 1 hour 30 minutes

INSTRUCTIONS

- Write your Name on each sheet of paper used or the front of the booklet used.
- Answer **all** the questions.
- You are permitted to use a graphical calculator in this paper.

INFORMATION

- The number of marks is given in brackets [] at the end of each question or part-question.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.
- Final answers should be given to a degree of accuracy appropriate to the context.
- The total number of marks for this paper is **72.**
- **You are reminded of the need for clear presentation in your answers.**

Section A (36 marks)

1 **(a)** You are given that $a = 0.2452$ and $b = 43.02$

(i) If the values of a and b are exact then find the values of $a + b$ and $a \times b$. [1]

(ii) If the values of a and b are accurate to 4 significant figures, find the values of $a + b$ and $a \times b$, giving your answers to the accuracy which is justified. Explain your reasoning. [4]

(b) The roots of the equation $x^2 - 50x + 1 = 0$ are α and β.

(i) Find α and β by the usual quadratic formula, giving the answers to 4 decimal places. [2]

(ii) The quadratic equation with roots α and β is $x^2 - (\alpha + \beta)x + \alpha\beta = 0$.

(A) Write down values of $\alpha + \beta$ and $\alpha \times \beta$. [1]

(B) Calculate these values from those obtained in part (i) and explain any discrepancies. [3]

2 The refractive index, r of a medium is determined by measuring two angles, α, and β, and using the formula

$$r = \frac{\sin \alpha}{\sin \beta}$$

(i) Find the calculated value of r if $\alpha = 37^{\circ}$ and $\beta = 31^{\circ}$. [1]

(ii) Calculate the interval within which the true value of r lies if the values of α and β are correct to the nearest degree. [3]

(iii) Hence

(A) determine the maximum possible magnitude of the relative error in the calculated value of r, [3]

(B) state the number of significant figures to which the value of r can be given with certainty. [1]

3 **(i)** Show that the equation $3\sin x - x = 0$ (where x is in radians) has a root in the interval [2, 3]. [2]

(ii) Show that the iterative formula $x_{r+1} = 3\sin x_r$ will not find this root. [2]

(iii) Use the bisection method to find the root to one decimal place. Show your working clearly. [3]

(iv) How many iterations of the bisection method would be required to give an error of less than 0.00005? [2]

4 The table below gives values for $f(x)$ for $x = 1.6$ to 2.4.

x	1.6	1.8	1.9	2.0	2.1	2.2	2.4
$f(x)$	0.078464	0.207688	0.299756	0.412	0.545824	0.702632	1.090816

(i) Obtain three estimates of $f'(2)$ using the central difference method with h taking values 0.4, 0.2 and 0.1. [3]

(ii) Show that, as h is halved, the differences between the estimates are approximately Divided by four. [1]

(iii) Hence obtain the best estimate you can of $f'(2)$. [2]

5 **(i)** Construct a quadratic polynomial passing through the following points. [2]

x	0.8	1.8	2.4
y	3.4	2.0	2.2

(ii) Use your polynomial to find the value of y when $x = 1.4$. [2]

(iii) Why is it not possible to use the Newton Interpolation Formula for this set of data? [1]

Section B (36 marks)

6 [Note: in this question, S_1 denotes the value obtained from a single application of Simpson's rule. A single application of Simpson's rule is often referred to as using two "strips". Similarly, S_2 denotes the value obtained when two applications of Simpson's rule are used to cover the range of integration.]

(i) Find, in terms of a the value of the integral $I = \int_0^a x^2 \, dx$. [1]

(ii) Hence find the error in each of the following approximations to I:

(A) T_1, the value given by a single application of the trapezium rule, [3]

(B) M_1, the value given by a single application of the mid-point rule. [3]

(iii) Find the value of S_1, where $S_1 = \frac{1}{3}\left(T_1 + 2M_1\right)$, and comment. [2]

(iv) For the integral $J = \int_0^1 \frac{1}{\sqrt{1+x^3}} \, dx$, find the values of T_1 and M_1; hence find an improved estimate, S_1. [4]

(v) Find also the values of the estimates T_2, M_2, S_2, where $S_2 = \frac{1}{3}\left(T_2 + 2M_2\right)$.

Hence give the value of J to the accuracy which is justified by your working. [5]

7 One step of the Newton-Raphson method for solving the equation f(x) = 0 can be expressed as

$$x_2 = x_1 - \frac{\mathrm{f}(x_1)}{\mathrm{f}'(x_1)} \qquad \text{(i)}$$

(i) Draw a sketch to illustrate the Newton-Raphson method. [2]

Now suppose that x_0 is near to x_1.

(ii) Write down an approximation to f '(x_1) based on x_0 and x_1. Substitute this approximation into (i) above and hence show that

$$x_2 \approx \frac{\mathrm{f}(x_0)x_1 - \mathrm{f}(x_1)x_0}{\mathrm{f}(x_0) - \mathrm{f}(x_1)}.$$ [4]

(iii) Let f(x) = x – cosx. Show that the equation f(x) = 0 has a root in the interval [0.5,1]. Use the iteration

$$x_{r+2} \approx \frac{\mathrm{f}(x_r)x_{r+1} - \mathrm{f}(x_{r+1})x_r}{\mathrm{f}(x_r) - \mathrm{f}(x_{r+1})}$$

to solve the equation, correct to 6 significant figures, taking $x_0 = 0.5$ and $x_1 = 1$. [6]

(iv) By considering the errors in x_2, x_3 and x_4, show that the convergence of this iteration appears to be better than first order. [A converging iteration is first order if the errors satisfy the relationship $\varepsilon_{r+1} \approx k\varepsilon_r$ for some constant k.] [5]

Qu		Answer	Mark	Comment
Section A				
1	**(a) (i)**	43.2652 and 10.548504	B1 **1**	
	(ii)	General rule is for adding, least no. of decimal places and multiplying least number of significant figures. Giving 43.27 and 10.55 with no certainty of the last digit so 43.3 and 10.6	B1 B1 B1 **3**	
	(b) (i)	$x=\frac{50\pm\sqrt{50^2-4}}{2}=\frac{50\pm 49.95998399}{2}$ $=49.9800$ and 0.0200	M1 A1 A1 **3**	
	(ii) (A)	50 and 1	B1 **1**	
	(B)	50 and 0.9996 The values in (A) are exact; these values have been computed from approximate values. The fact that 50 is the same is coincident.	B1 B1 B1 **3**	
2	**(i)**	r = 1.6849	B1 **1**	
	(ii)	Use 37.5 and 30.5 for upper bound = 1.19944 Use 36.5 and 31.5 for lower bound = 1.13842	M1 A1 A1 **3**	For both U.B. L.B.
	(iii) (A)	Max possible relative error = $\frac{1.16849-1.13842}{1.13842}=0.02649$ or = $\frac{1.16849-1.19944}{1.19944}=-0.02573$ $\Rightarrow$ maximum = 0.0265	M1 A1 A1 **3**	
	(iii) (B)	This means that r can be given to just one significant figure with certainty.	B1 **1**	
3	**(i)**	f(2) = 0.728 f(3) = –2.577 Because the curve is continuous and there is a sign change, there must be a root in this interval.	B1 B1 **2**	Both values Complete explanation
	(ii)	E.g. $x_0 = 2$, $x_1 = 3\sin 2 \Rightarrow x_1 = 2.728$ $\Rightarrow x_1 = 3\sin 2.728 \Rightarrow x_2 = 1.206$ Demonstrating divergence	M1 A1 **2**	x_0 = 2 or 3 Enough to show divergence

	(iii)	f(2.5) = –0.7 gives range [2, 2.5] f(2.25) = 0.08 gives range [2.25, 2.5] f(2.375) = -0.294 gives range [2.25,2.375] f(2.3125) = -0.1 gives range [2.25,2.3125] So to 1 decimal place, $x = 2.3$	M1 A1 A1 **3**	Trials At least two correct Answer
	(iv)	$\left(\frac{1}{2}\right)^n < 0.5\times10^{-4} \Rightarrow n\log 0.5 < \log 0.00005$ $\Rightarrow n > \frac{\log 0.00005}{\log 0.5} = 14.28$ i.e. 15 iterations	M1 A1 **2**	
4	**(i)**	$h = 0.4$ gives 1.26544 $h = 0.2$ gives 1.23736 $h = 0.1$ gives 1.23034	M1 A1 A1 **3**	Method One value All 3 values
	(ii)	Differences 0.4 to 0.2 = –0.028 0.2 to 0.1 = –0.007	B1 **1**	Ignore -ve sign here
	(iii)	$1.23034 - 0.007\left(\frac{1}{4}+\frac{1}{4^2}+......\right)$ $= 1.23034 - 0.007\left(\frac{\frac{1}{4}}{1-\frac{1}{4}}\right) = 1.22304$	M1 A1 **2**	But include here
5	**(i)**	$y = \frac{3.4(x-1.8)(x-2.4)}{(0.8-1.8)(0.8-2.4)} + \frac{2.0(x-0.8)(x-2.4)}{(1.8-0.8)(1.8-2.4)} + \frac{2.2(x-0.8)(x-1.8)}{(2.4-0.8)(2.4-1.8)}$ $\Rightarrow y = \frac{3.4(x-1.8)(x-2.4)}{(-1)(-1.6)} + \frac{2.0(x-0.8)(x-2.4)}{(1)(-0.6)} + \frac{2.2(x-0.8)(x-1.8)}{(1.6)(0.6)}$	M1 A1 **2**	Right form Correct algebra
	(ii)	When $x = 1.4$, $y = \frac{3.4(-0.4)(-1)}{(-1)(-1.6)} + \frac{2.0(0.6)(-1)}{(1)(-0.6)} + \frac{2.2(0.6)(-0.4)}{(1.6)(0.6)}$ $= 0.85 + 2.0 - 0.55$ $= 2.3$	M1 A1 **2**	Correct sub
	(iii)	Because the values of x are not equally spaced.	B1 **1**	

Section B				
6	(i)	$\int_0^a x^2 \, dx = \left[\frac{x^3}{3}\right]_0^a = \frac{a^3}{3}$		
	(ii) (A)	$T_1 = \frac{a}{2}\left(0+a^2\right) = \frac{a^3}{2}$: Error $= a^3\left(\frac{1}{2}-\frac{1}{3}\right) = \frac{a^3}{6}$	M1 A1 A1 **3**	Trap rule Value Error
	(ii) (B)	$M_1 = a\times\left(\frac{a}{2}\right)^2 = \frac{a^3}{4}$: Error $= a^3\left(\frac{1}{4}-\frac{1}{3}\right) = -\frac{a^3}{12}$	M1 A1 A1 **3**	
	(iii)	$S_1 = \frac{1}{3}\left(\frac{a^3}{2}+2\times\frac{a^3}{4}\right) = \frac{1}{3}a^3$ Simpson's rule is correct for a quadratic	B1 B1 **2**	
	(iv)	$T_1 = \frac{1}{2}\left(1+0.70711\right) = 0.85355$ $M_1 = 1\times\left(0.94281\right) = 0.94281$ $S_1 = \frac{1}{3}\left(0.85355+2\times0.94281\right) = 0.91305...$	B1 B1 B1 **3**	
	(v)	$T_2 = \frac{1}{2}\times\frac{1}{2}\left(1+2\times0.94281+0.70711\right) = 0.89818$ $M_2 = \frac{1}{2}\left(0.9923+0.8386\right) = 0.91545$ $S_2 = \frac{1}{3}\left(0.89818+2\times0.91545\right) = 0.90969$ $J = 0.91$ or $J = 0.910$	B1 B1 B1 B1 B1 **5**	Values
7	(i)	Adequate sketch showing a tangent And a second indicating how the next approximation is obtained	B1 B1 **2**	
	(ii)	$f'(x) = \frac{f(x_1)-f(x_0)}{x_1-x_0}$ $\Rightarrow x_2 \approx x_1 - \frac{f(x_1)}{\frac{f(x_1)-f(x_0)}{x_1-x_0}} = x_1 + \frac{f(x_1)\left(x_1-x_0\right)}{f(x_0)-f(x_1)}$ $\Rightarrow x_2 \approx \frac{x_1\left(f(x_0)-f(x_1)\right)+f(x_1)\left(x_1-x_0\right)}{f(x_0)-f(x_1)}$ $\approx \frac{x_1 f(x_0)-x_0 f(x_1)}{f(x_0)-f(x_1)}$	B1 M1 A1 A1 **4**	

	(iii)	f(0.5) = -0.38 < 0, f(1) = 0.46 > 0 n | x_r | f(x_r) | x_{r+1} 0 | 0.5 | -0.37758256 | | | 0.45969769 | 1 | 1 | 4 | 0.725481587 2 | 0.72548159 | -0.02269839 | 0.73839862 3 | 0.73839862 | -0.00114878 | 0.739087211 4 | 0.73908721 | 3.47711E-06 | 0.739085133 5 | 0.73908513 | -5.2727E-10 | 0.739085133 So root is x = 0.739085 to 6 significant figures.	B1 M1 A1 A1 A1 A1 **6**	Values Method x_2 x_3 x_4
	(iv)	$x_2 = 0.725482$: error $\approx 0.013603 = \varepsilon_2$ $x_3 = 0.738391$: error $\approx 0.000175 = \varepsilon_3$ $x_4 = 0.739087$: error $\approx 0.0000002 = \varepsilon_4$ $\frac{\varepsilon_3}{\varepsilon_2} = 0.051,\ \frac{\varepsilon_4}{\varepsilon_3} = 0.0029$ you would expect these to be approximately constant for 1st order convergence.	M1 A1 A1 M1 A1 **5**	 one error all 3