

MEI Structured Mathematics

Module Summary Sheets

Numerical Methods

(Version B—reference to new book)

Topic 1: Approximations

Topic 2: The solution of equations

Topic 3: Numerical integration

Topic 4: Approximating functions

Topic 5: Numerical differentiation

Topic 6: Rates of convergence in numerical processes

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References:
Chapter 1
Pages 1-6

Exercise 1A
Q. 6

Error

Let X be an approximation to an exact value, x , then the error is $e = X - x$ and hence $X = x + e$.

$$\text{The relative error is } r = \frac{e}{x} \Rightarrow X = x(1+r)$$

E.g. If a height of 182.3cm is recorded as 182 cm then the absolute error is 0.3 .

$$r = \frac{0.3}{182.3} \approx 0.00165$$

E.g. Find the absolute error when a mass of 69.7 kg is recorded to the nearest kg.

$$e = 69.7 - 70 = -0.3;$$

$$r = \frac{-0.3}{79.7} \approx -0.00376$$

References:
Chapter 1
Page 7

Exercise 1A
Q. 4

Recording numbers

Fixed point: decimal point in the correct position.
i.e. 145.64

Floating point: (also called standard form) in which the number is written as $a \times 10^b$ where $1 \leq a < 10$ and b is an integer.

E.g. Calculate the relative error in approximating

- (i) 0.123456 to 5 decimal places,
- (ii) 1234.56 to 3 significant places.

(i) To 5 d.p. number is 0.12346, an error of 0.000004

$$\text{Relative error is } \frac{0.00004}{1.23456} = 3.24 \times 10^{-5}$$

(ii) To 3 s.f. the number is 1230.

$$\text{Relative error is } \frac{4.56}{1234.56} = 3.69 \times 10^{-3}$$

References:
Chapter 1
Pages 10-11

Exercise 1B
Q. 3, 4

Interval estimates

If a number, n , is recorded to p decimal places then the maximum possible rounding error is . The interval within which the number lies is $[n \pm 0.5 \times 10^{-p}]$.

$$|\text{relative error}| \leq \left| \frac{5 \times 10^{-s} \times 10^b}{a \times 10^b} \right| \leq 5 \times 10^{-s}.$$

$$\text{E.g. } \frac{2}{3} = 0.6 \text{(chopping)}$$

$$\text{error} = 0.0\dot{6}, \text{ relative error} = \frac{0.0\dot{6}}{0.6} = 0.1$$

$$\text{E.g. } \frac{2}{3} = 0.7 \text{(rounding)}$$

$$\text{error} = 0.0\dot{3}, \text{ relative error} = \frac{0.0\dot{3}}{0.6} = 0.05$$

References:
Chapter 1
Pages 15-20

Exercise 1C
Q. 4

Exercise 1D
Q. 2, 5

Propagation of errors

1. The number of decimal places quoted in a sum should be no more than the number quoted in the term with the least number of decimal places.
2. The number of significant places in a product (or quotient) should be no more than the number of significant figures in the factor with the least significant figures.
3. Whenever two nearly numbers are subtracted then there may be a loss of many significant figures in the accuracy of the answer.

E.g.

$$1. 3.24 + 1.215 + 3.126 = 7.58 \text{ (2 d.p.)}$$

$$2. \frac{2.53 \times 0.0032}{4.326} = 0.0019 \text{ (correct to 2 s.f.)}$$

$$3. 2.5303 - 2.5300 = 0.003$$

(and this is correct to only 1 s.f.)

References:
Chapter 1
Pages 21-23

Ill-conditioned problems

If a small change to the input causes a large change in the output, then the problem is said to be ill-conditioned.

E.g. For $f(x) = \frac{1}{(x-1)}$;

$f(0.992) = -125$ but if the value 0.992 is rounded to 0.99 then $f(0.99) = -100$

Exercise 1D
Q.4

Numerical Methods

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Competence statements v1, 2, 3, 4, 5

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References:
Chapter 2
Pages 28-33

Exercise 2A
Q. 1, 3

References:
Chapter 2
Pages 33-36

Exercise 2B
Q. 2

References:
Chapter 2
Pages 38-40

Exercise 2C
Q. 3

References:
Chapter 2
Page 42-47

Exercise 2D
Q. 2

Initial Approximations

The process of finding the numerical solution of an equation is in two parts:

- the determination of an approximation to the roots;
- the successive improvement of these approximations to give the roots to any desired degree of accuracy.

The problem may suggest an approximate root. A **graphical calculator** can provide an approximate root, by altering ranges and using the zoom facility. A **table of values** may be calculated. If the graph is continuous and there is a change of sign between $x = a$ and $x = b$ then there is at least one root in the interval $[a, b]$.

Interval Bisection Method

If a root lies between $x = x_0$ and $x = x_1$ then $x_2 = \frac{1}{2}(x_0 + x_1)$

replaces either x_0 or x_1 so that the two values straddle the root.

The interval within which lies the root (the error bounds) is successively halved. At any stage the best possible estimate is the mid-point of the interval.

If the starting interval is $[x_0, x_1]$ then after n iterations the maximum possible error is $\frac{1}{2^{n+1}}(x_1 - x_0)$.

Fixed Point Iteration

Rearrange the equation $f(x) = 0$ in the form $x = g(x)$

Then apply the iterative procedure $x_{r+1} = g(x_r)$.

If a is the root and ϵ_r is the error in x_r then $x_r = a + \epsilon_r$.

$$x_{r+1} = g(x_r) = g(a + \epsilon_r) \approx g(a) + \epsilon_r g'(a)$$

$$\text{i.e. } a + \epsilon_{r+1} \approx g(a) + \epsilon_r g'(a).$$

Since $a = g(a)$ this gives $\epsilon_{r+1} \approx \epsilon_r g'(a)$

Given a good enough starting point, the procedure will converge if $|g'(x)| < 1$ for some interval including the root.

You need to be competent in the use of spreadsheets. Examples of each of these methods and all topics in this module can be worked on a spreadsheet.

Numerical Methods

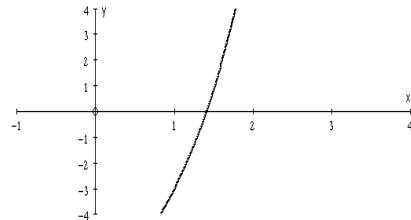
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Competence statements e1, e2, e3

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E.g. Solve the equation $x^3 + 3x - 7 = 0$

A graph indicates that there is one root in the interval $[1, 2]$.



$f(0) = -7$, $f(1) = -3$, $f(2) = 7$ demonstrating a sign change in the interval $[1, 2]$.

Interval bisection method

$f(0) = -7$, $f(1) = -3$, $f(2) = 7$; interval $[1, 2]$

So take $x_0 = 1$, $x_1 = 2$; $x_2 = \frac{1}{2}(1+2) = 1.5$

$f(1.5) = 0.875$; interval $[1, 1.5]$;

$$x_3 = \frac{1}{2}(1+1.5) = 1.25$$

$f(1.25) = -1.297$; interval $[1.25, 1.5]$;

$$x_4 = \frac{1}{2}(1.25+1.5) = 1.375$$

Continuing with the process will give an interval of $[1.40625, 1.408203]$, giving a value of 1.407 ± 0.001 to 3 decimal places.

Fixed Point Iteration method

$x^3 + 3x - 7 = 0$ may be rewritten $x = \frac{7}{x^2 + 3}$;

This is $x = g(x)$ where $g(x) = \frac{7}{x^2 + 3}$

Taking a near root to be $x_0 = 1.5$, $g'(x) = \frac{-14x}{(x^2 + 3)^2}$,

$g'(1.5) \approx \frac{21}{27.6} < 1$ so convergence will occur.

$$x_0 = 1.5 \Rightarrow x_1 = \frac{7}{1.5^2 + 3} = 1.333 \Rightarrow x_2 = \frac{7}{1.333^2 + 3} = 1.465$$

$$\Rightarrow x_3 = \frac{7}{1.465^2 + 3} = 1.3601 \Rightarrow x_4 = \frac{7}{1.3601^2 + 3} = 1.44$$

proceeding similarly gives $x_{15} = 1.40335$ and $x_{17} = 1.4044$

References:
Chapter 2
Pages 49-52

Exercise 2E
Q. 5

Newton-Raphson Method

The tangent at x_r to the curve will cut the x -axis at a point which is, generally, closer to the root than x_r .

$$x_{r+1} = x_r - \frac{f(x_r)}{f'(x_r)}$$

This is a rearrangement of the equation of the form $x_{r+1} = g(x_r)$ as above.

It can be shown that in this case $g'(\alpha) = 0$

It can also be shown that $\varepsilon_{r+1} \propto \varepsilon_r^2$.

This is called second order convergence.

If x_r is correct to p decimal places then x_{r+1} will be correct to approximately $2p$ decimal places.

References:
Chapter 2
Pages 52-54

Exercise 2E
Q. 9

Secant Method

This method approximates the derivative. So, instead of using the derivative in the Newton-Raphson formula the gradient of a chord is used.

$$\frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

approximates to the gradient of the curve at the point x_r . The iterative formula then becomes

$$x_{r+1} = x_r - \frac{x_{r-1} - x_r}{f(x_{r-1}) - f(x_r)} \times f(x_r) = \frac{x_r f(x_{r-1}) - x_{r-1} f(x_r)}{f(x_{r-1}) - f(x_r)}$$

Newton - Raphson method

$$x_{r+1} = x_r - \frac{f(x_r)}{f'(x_r)} \text{ with } f(x) = x^3 + 3x - 7$$

$$\Rightarrow f'(x) = 3x^2 + 3$$

$$\text{giving } x_{r+1} = x_r - \frac{x_r^3 + 3x_r - 7}{3x_r^2 + 3} \Rightarrow x_{r+1} = \frac{2x_r^3 + 7}{3x_r^2 + 3}$$

$$x_0 = 1 \Rightarrow x_1 = 1.5 \Rightarrow x_2 = 1.41025641$$

$$\Rightarrow x_3 = 1.406295005$$

$$\Rightarrow x_4 = 1.40628758$$

Secant method

$$x_{r+2} = x_{r+1} - \frac{x_{r+1} - x_r}{f(x_{r+1}) - f(x_r)} \cdot f(x_{r+1})$$

$$\text{with } f(x) = x^3 + 3x - 7$$

$$x_0 = 1, x_1 = 2 \Rightarrow f(x_0) = -3, f(x_1) = 7$$

$$\Rightarrow x_2 = 2 - \frac{2 - 1}{7 - (-3)} \cdot 7 = 2 - 0.7 = 1.3,$$

$$f(1.3) = -0.903$$

$$\Rightarrow x_3 = 1.3 - \frac{1.3 - 2}{-0.903 - 7} \cdot -0.903 = 1.3 + 0.08 = 1.38,$$

$$f(1.38) = -0.232$$

$$\Rightarrow x_4 = 1.38 - \frac{1.38 - 1.3}{-0.232 - (-0.903)} \cdot (-0.232) = 1.4077,$$

$$f(1.4077) = 0.01277$$

Continuing with the process gives $x = 1.406115$

References:
Chapter 2
Pages 57-59

Exercise 2F
Q. 1(iii)

False Position

(also called Linear Interpolation).

This method divides the interval between x_0 and x_1 , given $f(x_0) < 0$ and $f(x_1) > 0$ but takes account of the magnitude of $f(x_0)$ and $f(x_1)$.

The method uses similar triangles; note that the size of one side is $f(x_0)$.

$$x = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

Linear Interpolation method

Take $a = 1, b = 2, f(a) = -3, f(b) = 7$

$$x = \frac{7+6}{7+3} = 1.3, f(1.3) = -0.903$$

So take $a = 1.3, f(a) = -0.903, b = 2, f(b) = 7$

$$x = \frac{1.3 \times 7 + 2 \times 0.903}{7 + 0.903} = 1.38, f(1.38) = -0.232$$

So take $a = 1.38, f(a) = -0.232, b = 2, f(b) = 7$

$$x = \frac{1.38 \times 7 + 2 \times 0.232}{7 + 0.232} = 1.40, f(1.40) = -0.057$$

So take $a = 1.40, f(a) = -0.057, b = 2, f(b) = 7$

$$x = \frac{1.40 \times 7 + 2 \times 0.057}{7 + 0.057} = 1.4047, f(1.40) = -0.0139$$

Repeating the process will give $x = 1.4062$

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Competence statements e1, e2, e3, e4

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This has been obtained with a number of iterations until the values are consistent.

References:
Chapter 3
Pages 63-67

Exercise 3A
Q. 3

References:
Chapter 3
Pages 68-72

Exercise 3B
Q. 4

References:
Chapter 3
Pages 76-79

Exercise 3C
Q. 3

References:
Chapter 3
Pages 73-78

Exercise 3C
Q. 7

Mid-point Rule

The area under the curve is split into sections with equal widths.

The midpoints of these sections are taken and rectangles with these heights are used to replace the sections.

$$\text{Then } \int_a^b f(x) dx = h[f(m_1) + f(m_2) + \dots + f(m_n)] \\ = h \left[y_{\frac{1}{2}} + y_{\frac{3}{2}} + \dots + y_{\frac{n-1}{2}} \right]$$

Trapezium Rule

The area under the curve is split into trapezia with equal widths.

$$\text{Then } A = \frac{1}{2}h(y_0 + y_1) + \frac{1}{2}h(y_1 + y_2) + \dots + \frac{1}{2}h(y_{n-1} + y_n) \\ = \frac{1}{2}h(y_0 + 2(y_1 + y_2 + \dots + y_{n-1}) + y_n)$$

Simpson's Rule

Each strip is approximated by a parabola through three points on the curve. This method therefore requires an odd number of points.

Over the interval $[a, b]$ the area is divided into n strips of width $2h$. Then

$$A = \frac{1}{3}h(y_0 + 4y_1 + y_2) + \frac{1}{3}h(y_2 + 4y_3 + y_4) + \dots + \frac{1}{3}h(y_{2n-2} + 4y_{2n-1} + y_{2n}) \\ = \frac{1}{3}h(y_0 + 4y_1 + 2y_2 + \dots + 4y_{2n-1} + y_{2n}) \\ = \frac{1}{3}h(y_0 + y_{2n} + 4(\text{sum of odd ordinates}) + 2(\text{sum of even ordinates}))$$

The relationship between the methods

If T_n and T_{2n} are the values by the trapezium rule with n and $2n$ subintervals respectively, M_n the value using the midpoint rule with n subintervals, and S_n the value by using Simpson's rule with n applications, then

$$T_{2n} = \frac{1}{2}(T_n + M_n), \quad S_n = \frac{1}{3}(T_n + 2M_n)$$

Warning!

S_n is sometimes defined as Simpson's Rule applied with n strips and sometimes with n applications. So, for instance, S_2 can be defined as one application with 2 strips, or two applications, requiring 4 strips. In any examination question the definition will be explicitly given so that there is no confusion.

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Competence statements c3, c4

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Example: Find an estimate for

$$\int_0^1 \sqrt{1+x^2} dx$$

by the three methods with (i) 2 strips, (ii) 4 strips.

Values required

x	y	x	y
0	1	0.625	1.1792
0.125	1.0078	0.75	1.25
0.25	1.0308	0.875	1.3288
0.375	1.0680	1	1.4142
0.5	1.1180		

Midpoint method

$$(i) \text{ 2 strips: } A = 0.5[y_{0.25} + y_{0.75}] \\ = 1.1404$$

$$(ii) \text{ 4 strips: } A = 0.25[y_{0.125} + y_{0.375} + y_{0.625} + y_{0.875}] \\ = 1.1459$$

Trapezium Rule

$$(i) \text{ 2 strips: } I = \frac{h}{2}(y_0 + 2y_{0.5} + y_1) \\ = 1.1163$$

$$(ii) \text{ 4 strips: } I = \frac{h}{2}(y_0 + 2(y_{0.25} + y_{0.5} + y_{0.75}) + y_1) \\ = 1.151475$$

Simpson's Rule

$$(i) \text{ 1 application: } I = \frac{h}{3}(y_0 + 4y_{0.5} + y_1) \\ = 1.1477$$

$$(ii) \text{ 2 applications: } I = \frac{h}{3}(y_0 + 4(y_{0.25} + y_{0.75}) + 2y_{0.5} + y_1) \\ = 1.1478$$

Errors

For the Midpoint Rule and the Trapezium Rule

$$\varepsilon \propto h^2$$

For Simpson's Rule

$$\varepsilon \propto h^4$$

i.e. same number of steps means greater accuracy

Same accuracy means fewer steps.

References:
Chapter 4
Pages 83-89

Example 4.1
Page 87

Exercise 4A
Q. 4

References:
Chapter 4
Pages 90-94

Example 4.4
Page 93

Exercise 4B
Q.3, 6

Forward Difference method

For a set of equally spaced values x_0, x_1, x_2, \dots the forward difference operator Δ is defined by $\Delta f_r = f(x_{r+1}) - f(x_r)$, $\Delta^2 f_r = \Delta(\Delta f_r)$, etc.

For a polynomial of order n , $\Delta^n f(x_i) = \text{constant}$

A table of forward differences may be used:

- to interpolate values (i.e. values of $f(x)$ within the range of values given)
- to extrapolate values (i.e. values of $f(x)$ beyond those given)
- to find the gradient of the curve at a point.

E.g. Given the finite difference table for $y = f(x)$, find $f(4)$ by extrapolation. What assumptions must you make?

x	$f(x)$	Δf	$\Delta^2 f$	From the table it can be seen that 2nd differences are the same, so assume constant (i.e. $f(x)$ is a quadratic)
0	2		3	
1	5	3	9	6
2	14	9	15	6
3	29	15	21	6
4	50	21		$f(4) = 50$

The Newton Interpolation formula

The forward difference operator Δ is defined as

$$\Delta f_i = f_{i+1} - f_i.$$

$$\begin{aligned}\Delta f_0 &= f_1 - f_0 : f_1 = f_0 + \Delta f_0 \\ &\quad = (1 + \Delta)f_0 \\ \Delta f_1 &= f_2 - f_1 : f_2 = f_1 + \Delta f_1 \\ &\quad = (1 + \Delta)f_1 = (1 + \Delta)^2 f_0\end{aligned}$$

And in general $f_p = (1 + \Delta)^p f_0$

Using the Binomial expansion gives

$$\begin{aligned}f_p &= f(x_0 + ph) = (1 + \Delta)^p f_0 \\ &= f_0 + p\Delta f_0 + \frac{p(p-1)}{2!} \Delta^2 f_0 + \dots\end{aligned}$$

If $x = x_0 + ph$ then $p = \frac{x - x_0}{h}$, $p-1 = \frac{x - x_1}{h}$, etc.
giving

$$f(x) = f_0 + \frac{x - x_0}{h} \Delta f_0 + \frac{(x - x_0)(x - x_1)}{2!h^2} \Delta^2 f_0 + \dots$$

If the function is of degree n the expansion stops at $\Delta^n f_0$.

It is then usual to rearrange the polynomial in descending powers of x .

Note, however, that if interpolation is required (i.e. a value for $f(x)$ within the range of given values) this value for x may be substituted and the value for $f(x)$ found without the need for rearrangement.

Note that this method requires the given set of data to have equal intervals of x .

E.g. Given the following values of a function, find the value of $f(0.75)$.

x	0	0.5	1.0	1.5	2
$f(x)$	-1.000	1.125	4.000	8.375	1.500

For $f(0.75)$, $x = x_0 + ph$ with $x_0 = 0, x = 0.75$
and $h = 0.5 \Rightarrow p = 1.5$

$$\Rightarrow f(0.75) = f_0 + p\Delta f_0 + \frac{p(p-1)}{2} \Delta^2 f_0 + \dots$$

x	$f(x)$	Δf	$\Delta^2 f$	$\Delta^3 f$
0	-1			
0.5	1.125		0.75	
1.0	4.000	2.875		0.75
1.5	8.375	4.375	1.5	
2.0	15	6.625	2.25	

(N.B. The values in the $\Delta^3 f$ column being the same is an indication that the points of the function fit a cubic, though this does not prove that the function is a cubic.)

To find $f(0.75)$, $x = x_0 + ph$ with $x_0 = 0, x = 0.75$
and $h = 0.5 \Rightarrow p = 1.5$

$$\begin{aligned}\Rightarrow f(0.75) &= -1 + 1.5 \times 2.125 + \frac{1.5 \times 0.5}{2} - 0.75 + \frac{1.5 \times 0.5 \times -0.5}{6} 0.75 \\ &= -1 + 3.1875 + 0.28125 - 0.046875 \\ &= 2.421875\end{aligned}$$

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Competence statements: f1

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References:
Chapter 4
Page 94

The Lagrange Interpolating polynomial

The Lagrange method is an alternative method to find a polynomial that fits a set of data.
It has the advantage that the set of data given do not have to be at equally spaced intervals.

The straight line through the points (a, A) and (b, B) has equation $y = \frac{A(x-b)}{(a-b)} + \frac{B(x-a)}{(b-a)}$.

The quadratic function whose graph passes through the points (a, A) , (b, B) and (c, C) has equation

$$y = \frac{A(x-b)(x-c)}{(a-b)(a-c)} + \frac{B(x-c)(x-a)}{(b-c)(b-a)} + \frac{C(x-a)(x-b)}{(c-a)(c-b)}$$

Similarly a cubic passing through four points would have four terms.

This form may be quite easily written down. It only needs to be simplified to give the polynomial in descending powers in an examination if that is specifically demanded.

If you are asked find a value of $f(x)$ you will not be required to simplify the polynomial.

Exercise 4B
Q. 47

E.g. Find the quadratic polynomial passing through the points $(0, -3)$, $(1, -1)$ and $(3, 9)$.

The equation is

$$\begin{aligned} y &= \frac{(-3)(x-1)(x-3)}{(0-1)(0-3)} + \frac{(-1)(x-3)(x-0)}{(1-3)(1-0)} \\ &\quad + \frac{9(x-0)(x-1)}{(3-0)(3-1)} \\ \Rightarrow y &= -\frac{(x-1)(x-3)}{1} + \frac{x(x-3)}{2} + \frac{9x(x-1)}{3 \times 2} \\ \Rightarrow y &= -(x-1)(x-3) + \frac{1}{2}x(x-3) + \frac{3}{2}x(x-1) \\ \Rightarrow y &= -x^2 + 4x - 3 + \frac{1}{2}x^2 - \frac{3}{2}x + \frac{3}{2}x^2 - \frac{3}{2}x \\ \Rightarrow y &= x^2 + x - 3 \end{aligned}$$

References:
Chapter 4
Pages 96-100

Truncating Newton's Interpolating polynomial

If data entries in the table are approximations then no difference column will be constant. If you look at the table then it should be possible to find the first column that is *nearly* constant, meaning that the next difference column entries will be very small. An approximating polynomial can then be constructed.

Exercise 4C
Q.4, 5

E.g. An experiment yields the following values of two variables, x and y . Estimate a value for y when $x = 1.5$.

x	1.1	1.3	1.6
y	6.4	7.3	8.8

The equation is

$$\begin{aligned} y &= \frac{6.4(x-1.3)(x-1.6)}{(1.1-1.3)(1.1-1.6)} + \frac{7.3(x-1.1)(x-1.6)}{(1.3-1.1)(1.3-1.6)} \\ &\quad + \frac{8.8(x-1.1)(x-1.3)}{(1.6-1.1)(1.6-1.3)} \end{aligned}$$

When $x = 1.5$

$$\begin{aligned} y &= \frac{6.4(1.5-1.3)(1.5-1.6)}{(1.1-1.3)(1.1-1.6)} + \frac{7.3(1.5-1.1)(1.5-1.6)}{(1.3-1.1)(1.3-1.6)} \\ &\quad + \frac{8.8(1.5-1.1)(1.5-1.3)}{(1.6-1.1)(1.6-1.3)} \\ \Rightarrow y &= \frac{6.4(0.2)(-0.1)}{(-0.2)(-0.5)} + \frac{7.3(0.4)(-0.1)}{(0.2)(-0.3)} + \frac{8.8(0.4)(0.2)}{(0.5)(0.3)} \\ \Rightarrow y &= -1.28 + 4.87 + 4.69 = 8.28 \end{aligned}$$

E.g. Use the data in the table on the previous page to find the cubic polynomial which fits the data.

$$h = 0.5, x_0 = 0, x_1 = 0.5, x_2 = 1.0$$

$$f_0 = -1, \Delta f_0 = 2.125, \Delta^2 f_0 = 0.75, \Delta^3 f_0 = 0.75$$

$$\begin{aligned} \Rightarrow f(x) &= -1 + \frac{x}{0.5} \times 2.125 + \frac{x(x-0.5)}{2 \times 0.5^2} \times 0.75 \\ &\quad + \frac{x(x-0.5)(x-1.0)}{6 \times 0.5^3} \times 0.75 \\ &= -1 + 4.25x + 1.5x(x-0.5) + x(x-0.5)(x-1) \\ &= -1 + 4.25x + 1.5x^2 - 0.75x + x^3 - 1.5x^2 + 0.5x \\ &= x^3 + 4x - 1 \end{aligned}$$

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Competence statements: f1, f2

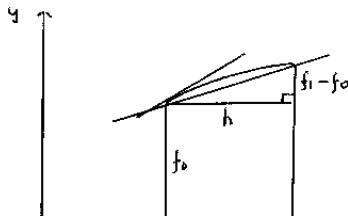
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References:
Chapter 5
Pages 103-105

Example 5.1
Page 104

Numerical Differentiation

Given two points on a curve, x_0 and x_1 where $x_1 = x_0 + h$, then the gradient of the tangent at x_0 may be approximated by the gradient of the chord.



$$\text{Gradient} = \frac{f_1 - f_0}{x_1 - x_0} = \frac{\Delta f_0}{h} \quad \text{i.e. } f'(x) \approx \frac{\Delta f_0}{h}$$

In practice, you would take a sequence of values of h and look for convergence in the values of f' . As h gets small this formula will become very inaccurate.

References:
Chapter 5
Pages 105-107

Example 5.3
Page 106

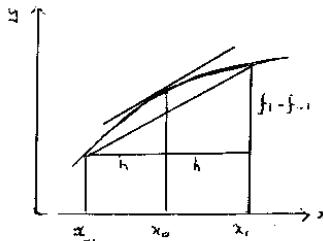
Exercise 5A
Q. 1

References:
Chapter 5
Pages 108-109

Exercise 5A
Q. 3

Central Difference Method

A better approximation for the gradient at x_0 would be to take values on both sides of x_0 .



$$\text{Gradient} = \frac{f_1 - f_{-1}}{x_1 - x_{-1}} = \frac{f(x+h) - f(x-h)}{2h}$$

$$\text{i.e. } f'(x) \approx \frac{f(x+h) - f(x-h)}{2h}$$

As above, in practice, you would take a sequence of values of h and look for convergence in the values of f' . As h gets very small this formula will become inaccurate.

Calculating the error in $f(x)$ when there is an error in x .

If there is an error of h in the value of x then $X = x + h$ and

$$f(X) - f(x) \approx f'(x)h$$

i.e. if the error in x is h then the error in $f(x)$ is approximately $f'(x)h$.

Numerical Methods

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Competence statements: f1

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E.g. Obtain an approximation to the derivative

$$\text{of } f(x) = \tan x \text{ at } x = \frac{\pi}{4}.$$

With $h = 0.1$,

$$\tan\left(\frac{\pi}{4} + 0.1\right) = 1.223, \tan\left(\frac{\pi}{4}\right) = 1$$

$$\Rightarrow f'(x) \approx \frac{1.223 - 1}{0.1} = 2.23$$

With $h = 0.05$,

$$\tan\left(\frac{\pi}{4} + 0.05\right) = 1.105, \tan\left(\frac{\pi}{4}\right) = 1$$

$$\Rightarrow f'(x) \approx \frac{1.105 - 1}{0.05} = 2.107$$

E.g. Obtain an approximation to the derivative

$$\text{of } f(x) = \tan x \text{ at } x = \frac{\pi}{4}.$$

With $h = 0.1$,

$$\tan\left(\frac{\pi}{4} + 0.1\right) = 1.223, \tan\left(\frac{\pi}{4} - 0.1\right) = 0.818$$

$$\Rightarrow f'(x) \approx \frac{1.223 - 0.818}{2 \times 0.1} = 2.025$$

With $h = 0.05$,

$$\tan\left(\frac{\pi}{4} + 0.05\right) = 1.105, \tan\left(\frac{\pi}{4} - 0.05\right) = 0.905$$

$$\Rightarrow f'(x) \approx \frac{1.105 - 0.905}{2 \times 0.05} = 2$$

E.g. Obtain an approximation to the derivative of $f(x)$ by the forward difference and central difference methods for $x = 0.2, 0.3$ and 0.4 , given the values of the function in the table.
($h = 0.1$)

x	y	F.D.	C.D.	dy/dx
0	1			0
0.1	1.005	0.05	0.1	0.0995037
0.2	1.02	0.15	0.195	0.1961161
0.3	1.044	0.24	0.285	0.2873479
0.4	1.077	0.33	0.37	0.3713907
0.5	1.118	0.41		0.4472136

In fact the function is $y = \sqrt{x^2 + 1}$ which can be differentiated by techniques in C3 to give

$$\frac{dy}{dx} = \frac{x}{\sqrt{x^2 + 1}}$$

giving the values in the last column for comparison.

Summary NM Topic 6: Rates of convergence in numerical processes

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References:
Chapter 6
Pages 112-120

Example 6.1
Page 117

Exercise 6A
Q. 2, 4(i)

References:
Chapter 6
Pages 121-127

Example 6.5
Page 127

Exercise 6B
Q. 1, 3

References:
Chapter 6
Pages 128-130

Exercise 6C
Q. 1, 3

Rates of convergence of sequences

A sequence has *first order convergence* if

$$\varepsilon_{r+1} \propto \varepsilon_r \text{ i.e. } \varepsilon_{r+1} = k\varepsilon_r$$

for all r where $|k| < 1$.

A sequence has *second order convergence* if

$$\varepsilon_{r+1} \propto \varepsilon_r^2 \text{ i.e. } \varepsilon_{r+1} = k\varepsilon_r^2$$

for all r . We also require $|k\varepsilon_r| < 1$.

ε_r is the error in the r th term. It is not possible to calculate this error as we need the exact value (for which we are looking!) to calculate it.

We often take the most accurate value we have to be the exact value for the purposes of calculating the errors. Alternatively we consider the ratio of differences of iterated values.

The *ratio of differences* for a sequence = $\frac{x_{r+2} - x_{r+1}}{x_{r+1} - x_r}$

For a first order converging sequence, the ratio of differences will be equal to k .

For a second order converging sequence the ratio requires the use of the limit, which for a numerical process is the best you can do. Then the ratio of the error in one step with the square of the error at the previous step

Rates of convergence in numerical integration

Midpoint rule: The absolute error is proportional to h^2 . This is a second order method.

Simpson's rule: The absolute error is proportional to h^4 . This is a fourth order method.

Extrapolated estimates

$$T = T_{2n} + \frac{T_{2n} - T_n}{3} = \frac{4T_{2n} - T_n}{3}$$

$$S = S_{2n} + \frac{S_{2n} - S_n}{15} = \frac{16S_{2n} - S_n}{15}$$

Rates of convergence in numerical differentiation

- For forward difference approximations, the error is proportional to h . It is a first order method. The ratio of differences between estimates obtained by repeatedly halving h is around 0.5
- For central difference approximations the error is proportional to h^2 . It is a second order method. The ratio of differences when h is halved is around 0.25

Numerical Methods

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Competence statements: e4, c3, c4

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E.g. to solve the equation $x^3 + 2x - 9 = 0$.

$f'(x) > 0$ for all x so only one root.

$f(1) = -6, f(2) = 3$ so root in interval [1, 2].

(i) Fixed point iteration:

Use the iterative formula $x_{r+1} = \sqrt[3]{9 - 2x_r}$ with $x_0 = 1$

A spreadsheet tabulation gives the following:

x_r	x_{r+1}	$x_r - x_{r+1}$	$(x_{r+1} - x_r)/(x_r - x_{r-1})$
1	1.912931	-0.91293	
1.912931	1.729601	0.18333	-4.97972
1.729601	1.769528	-0.03993	-4.59166
1.769528	1.760986	0.008542	-4.67421
1.760986	1.76282	-0.00183	-4.65645
1.76282	1.762427	0.000394	-4.66026
1.762427	1.762511	-8.4E-05	-4.65945
1.762511	1.762493	1.81E-05	-4.65962
1.762493	1.762497	-3.9E-06	-4.65958
1.762497	1.762496	8.35E-07	-4.65959

from which it can be seen that the ratio of differences is approximately constant.

(ii) The Newton-Raphson iterative formula gives

$$x_{r+1} = \frac{2x_r^3 + 9}{3x_r^2 + 2} \text{ with } x_0 = 1$$

A spreadsheet tabulation gives the following:

x_r	x_{r+1}	$x_r - \alpha$	$(x_{r+1} - \alpha)/(x_r - \alpha)$
1	2.2	-0.762496376	
2.2	1.833898305	0.437503624	0.752499
1.83389831	1.764786364	0.071401929	0.373032
1.76478636	1.762498823	0.002289987	0.449173
1.76249882	1.762496376	2.44651E-06	0.466532
1.76249638	1.762496376	2.79621E-12	0.467169

thus demonstrating second order convergence.

E.g. Values of $f(x)$ are given in the table below.

Using these values, estimate $\int_0^1 f(x) dx$ using extrapolation.

$$x \quad 0 \quad 0.25 \quad 0.5 \quad 0.75 \quad 1 \\ y \quad 1 \quad 0.89443 \quad 0.81649 \quad 0.75593 \quad 0.70711$$

$$T_1 = \frac{1}{2}(1 + 0.70711) = 0.85355$$

$$\text{Similarly } T_2 = 0.83502, \quad T_4 = 0.83010$$

$$\Rightarrow T = \frac{4T_4 - T_2}{3} = 0.82846$$

$$\text{Similarly } S_1 = 1.65769, S_2 = 1.65692$$

$$\Rightarrow S = \frac{16S_2 - S_1}{15} = 1.65687$$