## MEI Structured Mathematics

## Module Summary Sheets

## Mechanics 3 (Version B: Reference to new book)

Topic 1: Circular motion
Topic 2: Elastic springs and strings
Topic 3: Modelling oscillations
Topic 4: Volumes of revolution and centres of mass by integration

Topic 5: Dimensions and units

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| Chapter 1 |
| Pages 1-3 |

## Example 1.1

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## Exercise 1A

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References:
Chapter 1
Pages 7, 8

Example 1.2
Page 8

## Angular Speed and Velocities

A force is required to keep a body moving in a circle as its velocity is always changing.
If a particle at P is moving in a circle then the angle of OP with any fixed direction changes with time.

$\frac{\mathrm{d} \theta}{\mathrm{d} t}$ is written as $\dot{\theta}$ or $\omega$

The position is:

$$
x=r \cos \theta, y=r \sin \theta
$$

Velocity: tangential component: $v=r \frac{\mathrm{~d} \theta}{\mathrm{~d} t}=r \dot{\theta}=r \omega$ radial component: 0
Acceleration : tangential component : $r \ddot{\theta}=r \dot{\omega}$

$$
\text { radial component: }-r \dot{\theta}^{2}=-r \omega^{2}=-\frac{v^{2}}{r}
$$

the positive tangential $\quad \stackrel{\dot{\theta}}{ }=r_{\omega}$ direction


## Acceleration

References:
Chapter 1
Pages 9-13

Example 1.4
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Exercise 1B Q. 2

## Forces required for circular motion

Newton's 2nd Law ( $F=m a$ ) may be applied in the tangential and radial directions.
Tangential component is $m r \dot{\theta}$
Radial component is $-m r \dot{\theta}^{2}=-m r \omega^{2}=-\frac{m v^{2}}{r}$
Mechanics 3
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Competence statements r $1,2,3,4,5,6,7,8$ © MEI
E.g. Car A travels round a circle of radius 30 m at $33 \mathrm{~m} \mathrm{~s}^{-1}$. Car B travels round a circle of radius 25 m at $1.2 \mathrm{rad} \mathrm{sec}^{-1}$. Which car is moving the faster?
If linear speeds are compared,
Car B: $v=r \omega=1.2 \times 2.5=30 \mathrm{~m} \mathrm{~s}^{-1}$
so car A has greater linear speed.
If angular speeds are compared,
Car A: $v=33, r=30 \Rightarrow \omega=\frac{33}{30}=1.1 \mathrm{rad} \mathrm{s}^{-1}$
So B has greater angular speed.
E.g. The minute hand of my watch is 2 cm long. What is the speed of the tip of the hand?

$$
\begin{aligned}
& r=0.02 \mathrm{~m}, \omega=2 \pi \text { rads per hour }=\frac{2 \pi}{3600} \text { rads s }^{-1} \\
& \Rightarrow v=\frac{0.02 \times 2 \pi}{3600} \mathrm{~m} \mathrm{~s}^{-1} \approx 3.5 \times 10^{-5} \mathrm{~m} \mathrm{~s}^{-1}(2 \text { s.f. })
\end{aligned}
$$

E.g. What is the speed of the second hand, which is 2.1 cm long?

$$
v=3.5 \times 10^{-5} \times 60 \times \frac{2.1}{2} \mathrm{~m} \mathrm{~s}^{-1} \approx 2.2 \times 10^{-3} \mathrm{~m} \mathrm{~s}^{-1}(2 \text { s.f. })
$$

E.g. A horizontal, circular bend on a railway track has a radius of 500 m and a train of mass $m \mathrm{~kg}$ negotiates it at 60 mph . What is the lateral force, $R \mathrm{~N}$, between its wheels and the rails in terms of $m$ ?

$$
\begin{aligned}
& 60 \mathrm{mph} \approx 26.8 \mathrm{~m} \mathrm{~s}^{-1} \\
& R=m a=m \frac{v^{2}}{r}=\frac{26.8^{2}}{500}=1.44 m
\end{aligned}
$$

E.g. One end of a light inextensible string of length 0.8 m is attached to a point V . The other end is attached to the point O which is 0.4 m vertically below $P$. A small smooth bead, $B$, of mass 0.02 kg is threaded on the string and moves in a horizontal circle, with centre O and radius 0.3 m . OB rotates with constant angular speed $\omega \mathrm{rad} \mathrm{s}^{-1 .}$
Find the tension in the string and $\omega$.
Res vert. $\quad T \cos \theta-m g=0$

$\Rightarrow T=0.02 g \times \frac{5}{4}=0.025 g(=0.245 \ldots)$
Res. Hor. $T+T \sin \theta=m r \omega^{2} \Rightarrow \frac{8}{5} T=m r \omega^{2}$
$\Rightarrow \omega^{2}=\frac{8 \times 0.025 \mathrm{~g}}{5 \times 0.02 \times 0.3}=65.33 \ldots$
$\Rightarrow \omega=8.1$ (2 s.f.)

| References: |
| :---: |
| Chapter 1 |
| Pages 15-19 |

## Example 1.6

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References:
Chapter 1
Pages 26-28

Example 1.7
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Exercise 1C
Q. 3

## References:

Chapter 1
Pages 30-35

Example 1.8
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Exercise 1D
Q. 4

Banked Tracks - for motion in horizontal circle When there is "perfect banking" there will be no friction or lateral force due to rails, etc., so the only horizontal force acting will be the resolved part of the normal reaction, $R$. - i.e. $F=0$


Given that $F=0$,
$R \sin \theta=\frac{m v^{2}}{r}$
where $R \cos \theta=m g$
$\Rightarrow g \tan \theta=\frac{v^{2}}{r}$
If the angle of the banking is greater or less than that for perfect banking, then the body will stay on the circular track only if there is friction or lateral force. If there is friction (i.e. a road rather than rails) then it may be enough to hold the body on that circular motion or it may move to a circular motion with a greater or lesser radius.

## Circular motion with constant angular acceleration

The equations for motion with constant angular acceleration $\alpha$ are

$$
\begin{array}{ll}
\omega=\omega_{0}+\alpha t & \omega^{2}=\omega_{0}^{2}+2 \alpha \theta \\
\theta=\omega_{0} t+\frac{1}{2} \alpha t^{2} & \theta=\frac{1}{2}\left(\omega_{0}+\omega\right) t
\end{array}
$$

where $\omega_{0}$ is the initial angular speed.

## Motion in a vertical circle

If a particle is constrained to move in a vertical circle (i.e. at the end of a rigid rod or on a wire)
Using conservation of mechanical energy:
KE + GPE = const;
If we take GPE $=0$ at the level of O


Since energy is conserved:

$$
\begin{aligned}
& \frac{1}{2} m u^{2}-m g r=\frac{1}{2} m v^{2}-m g r \cos \theta \\
& \Rightarrow v^{2}=u^{2}-2 g r(1-\cos \theta)
\end{aligned}
$$

## Mechanics 3

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Competence statements $\mathrm{r} 1,2,3,4,5,6,7,8$
(C) MEI
E.g. A car of mass 1200 kg is moving horizontally at constant speed round a banked track which is a circle of radius 50 m . The bank is angled at $12^{0}$.
Find the force on the wheels if the speed is
(i) $10.2 \mathrm{~m} \mathrm{~s}^{-1}$, (ii) $13 \mathrm{~m} \mathrm{~s}^{-1}$.
(The diagram is opposite)
No vertical acceleration $\Rightarrow R \cos \theta=m g+F \sin \theta$
Horizontally: $\frac{m v^{2}}{r}=R \sin \theta+F \cos \theta$
$\Rightarrow \frac{m v^{2}}{r}=\frac{(m g+F \sin \theta) \sin \theta}{\cos \theta}+F \cos \theta$
$=\frac{m g \sin \theta+F\left(\sin ^{2} \theta+\cos ^{2} \theta\right)}{\cos \theta}$
$=\frac{m g \sin \theta+F}{\cos \theta} \Rightarrow F=\frac{m v^{2} \cos \theta}{r}-m g \sin \theta$
(i) $v=10.2, r=50, \theta=12$
$\Rightarrow F=m(2.035-2.035)=0$
(ii) $v=13, r=50, \theta=12$
$\Rightarrow F=m(3.306-2.037)=1522 \mathrm{~N}$
N.B. You do not need to know the positive direction of $F$ before resolving. If $F$ had been taken positive up the plane you would simply have obtained the answer -1522 N .
E.g. A particle is swung in a vertical circle at the end of a light rod of radius $r$ with initial speed $u$ at the bottom of the circle. Find the condition for the particle to complete the circle.
The particle completes the circle if $v>0$ when $\theta=180^{\circ}$
i.e. $\cos \theta=-1$

Given $v^{2}=u^{2}-2 r g(1-\cos \theta)$ then if $v>0$ and $\cos \theta=-1$, $\frac{u^{2}}{r}>2 g(1-\cos \theta)=4 g$ so $u^{2}>4 r g$
E.g. A small ring of mass $m$ is threaded onto a smooth vertical circular wire of radius 0.15 m . The ring is projected from the lowest point of the wire with speed $2.1 \mathrm{~m} \mathrm{~s}^{-1}$. How far above the bottom of the wire does the ring reach?
The ring stops when $v=0$ and the angle is $\theta$,
Energy at $\mathrm{A}=\frac{1}{2} m u^{2}-m g \times 0.15$
Energy when ring stops $=0-m g \times 0.15 \times \cos \theta$
Referred to the diagram opposite and conserving energy:
$\frac{1}{2} m u^{2}-m g \times 0.15=-m g \times 0.15 \cos \theta$
$\Rightarrow \cos \theta=-\frac{1}{2} \times \frac{2.1^{2} m}{0.15 m g}+1=-0.5 \Rightarrow \theta=120^{\circ}$
So the height reached is $0.15-0.15 \cos 120^{\circ}=0.225 \mathrm{~m}$

| References: <br> Chapter 1 <br> Pages 36-37 |
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References: Chapter 1 Page 38

## References:

Chapter 1
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Example 1.10 Page 40

Exercise 1D Q. 5, 7

When vertical circular motion breaks down (i.e. when the particle may leave the circular path)

- a particle on a light, inextensible string; no air resistance

N2L towards the centre gives
$T+m g \cos \theta=m r \omega^{2}$
The particle will remain moving in the circle if $T \geq 0$
$\Rightarrow \cos \theta \leq \frac{r \omega^{2}}{g}$
If this is so when $\cos \theta=1$


Fig. 1
(i.e at the top, with the maximum value for $\cos \theta$ ) then $r \omega^{2} \geq g$ or $v^{2} \geq r g$ where $v$ is the least speed and $\omega$ is the least angular velocity.

## -motion on the inside of a vertical circle; no friction

N2L towards the centre gives
$\mathrm{R}+m g \cos \theta=m r \omega^{2}$
The particle will remain
moving in the circle if $\mathrm{R} \geq 0$
We require $\cos \theta \leq \frac{r \omega^{2}}{g}$
If this is so when $\cos \theta=1$


Fig. 2
(i.e. at the top, with the maxiumum value for $\cos \theta$ ) then $r \omega^{2} \geq g$ or $v^{2} \geq r g$ where $v$ is the least speed and $\omega$ is the least angular velocity.

## -motion on the outside of a circle; no friction

N 2 L towards the centre gives
$m g \cos \theta-\mathrm{R}=m r \omega^{2}$
The particle will remain on
the surface of the circle
if $R \geq 0$
We require $\cos \theta \geq \frac{r \omega^{2}}{g}$

or $\frac{v^{2}}{r g} \leq \cos \theta$ where $v$ is the speed when the angle is $\theta$.

## Mechanics 3

Version B: page 4
Competence statements r $1,2,3,4,5,6,7,8$
(C) MEI
E.g. A particle is attached to the end of a light inextensible string of length $r$ and swung in a vertical circle with no air resistance. Find the least speed at the bottom of the circle for the particle to complete circular motion.
Using the notation of Fig. 1 and the work opposite; If the speed at the bottom is $u$ and that at the top is $v$ then the conservation of energy gives
$\frac{1}{2} m u^{2}=\frac{1}{2} m v^{2}+2 m g r$ so $v^{2}=u^{2}-4 r g$
Using the result that $v^{2} \geq r g$ derived opposite we require $u^{2}-4 r g \geq r g \Rightarrow u^{2} \geq 5 r g$
E.g. A particle is projected up the inside of a smooth sphere of radius $r$. Find the least angular speed at the bottom of the circle for the particle to complete circular motion.
Using the notation of Fig. 2 and the work opposite; If the angular speed at the bottom is $\Omega$ and that at the top is $\omega$ then the conservation of energy gives
$\frac{1}{2} m r^{2} \Omega^{2}=\frac{1}{2} m r^{2} \omega^{2}+2 m g r$ so $\omega^{2}=\Omega^{2}-\frac{4 g}{r}$
Using the result that $r \omega^{2} \geq g$ derived opposite
we require $r\left(\Omega^{2}-\frac{4 g}{r}\right) \geq g \Rightarrow \Omega^{2} \geq \frac{5 g}{r}$
Note that the solution is very similar to that above. All that has been done is to replace $u$ by $r \Omega$ and $v$ with $r \omega$.
E.g. A particle of mass 0.05 kg is gently displaced from rest from the top of a solid smooth sphere of radius 0.5 m . Find where it leaves the surface.
Refer to Fig 3.
N2L towards the centre gives $m g \cos \theta-R=0.05 \times 0.5 \times \omega^{2}$ If $\theta_{1}$ is the angle where the particle leaves the surface then $R=0$ and at this point $0.05 g \cos \theta_{1}=0.05 \times 0.5 \times \omega^{2}$

$$
\text { so } \omega^{2}=2 g \cos \theta_{1}
$$

Loss of GPE is $m g h=m g r\left(1-\cos \theta_{1}\right)$
Gain in KE is $\frac{1}{2} m v^{2}=\frac{1}{2} m(r \omega)^{2}$
Conserving energy gives $\frac{1}{2} m r^{2} \omega^{2}=m g r\left(1-\cos \theta_{1}\right)$
so $\omega^{2}=\frac{2 g r\left(1-\cos \theta_{1}\right)}{r^{2}}=4 g\left(1-\cos \theta_{1}\right)$
But $\omega^{2}=2 g \cos \theta_{1}$ so $4 g\left(1-\cos \theta_{1}\right)=2 g \cos \theta_{1}$
$\Rightarrow 6 \cos \theta_{1}=4 \Rightarrow \cos \theta=\frac{2}{3}$
$\Rightarrow \theta=48.2^{0}$ (3 s.f.)

References:
Chapter 2
Pages 51-56

Example 2.1
Page 55

Exercise 2A
Q. 2, 4

Example 2.3
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Exercise 2B
Q. 7

## References: <br> Chapter 2 <br> Pages 66-70

Example 2.5
Page 68

Exercise 2C Q. 4

## References: <br> Chapter 2 <br> Pages 74-75

Example 2.6
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Exercise 2D
Q. 4

## Strings and springs

We sometimes model a string to be inextensible. This is not always the case: strings and springs which stretch are said to be elastic.
The length when there is no force attached to it is called the natural length. When stretched the increase in length is called the extension. A spring may be compressed. A string cannot be compressed.

## Hooke's Law

The tension in an elastic spring is proportional to the extension. If a spring is compressed the thrust is proportional to the decrease in length of the spring.
If the natural length is $l_{0}$ and the extension is $x$, resulting from a force $T$, then

$$
T=k x .
$$

where $k$ is the stiffness.

Or we write $T=\frac{\lambda}{l_{0}} x$
where $\lambda$ is called the modulus of elasticity.

## Work and Energy

The total work done in stretching a string by $x$ from its natural length $l_{0}$ is given by

$$
\text { Work done }=\frac{1}{2} k x^{2}=\frac{\lambda}{2 l_{0}} x^{2} .
$$

This is known as the elastic potential energy. (E.P.E.)

## The Principle of Conservation of Energy

 The principle of conservation of energy may be used when the energy changes only involve (some of) KE, GPE and EPE.
## Vertical motion

When a particle attached to a spring is released
from a point other than the equilibrium position then the principles of conservation of energy may be applied.
Conservation of energy gives:
Increase in KE + increase in GPE + increase in $\mathrm{EPE}=0$
Care must be taken when an elastic string is raised above its equilibrium position as it will go slack rather than compress.

## Mechanics 3

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Competence statements h1, 2, 3, 4, 5 © MEI
E.g. A piece of light string is 50 cm long.

Experiments have indicated that $\lambda=80 \mathrm{~N}$. A mass of 5 kg is hung from the string. Find the extension.
$T=5 g=\frac{\lambda}{l_{0}} x \Rightarrow x=\frac{5 g \times 0.5}{80}=0.3062$
Extension is 30.6 cm (3 s.f.).
E.g. A light spring of natural length 50 cm is attached to a ceiling. When a mass of 4 g is hung in equilibrium, the extension is 5 cm . Find
(i) the tension in the spring,
(ii) the stiffness of the spring,
(iii) the modulus of elasticity of the spring. Take $g=10 \mathrm{~m} \mathrm{~s}^{-2}$.
(i) $T=m g=0.004 g=0.04 \mathrm{~N}$
(ii) $T=k x \Rightarrow 0.04=0.05 k \Rightarrow k=0.8 \mathrm{~N} \mathrm{~m}^{-1}$
(iii) $T=\frac{\lambda}{x_{0}} x \Rightarrow \lambda=T \frac{x_{0}}{x}=\frac{50}{5} \times 0.04=0.4 \mathrm{~N}$
E.g. Find the EPE stored in the example above when the system is in equilibrium..

$$
\frac{1}{2} k x^{2}=\frac{1}{2} \times 0.8 \times 0.05^{-2}=1 \times 10^{-3}=0.001 \mathrm{~J}
$$

E.g. Suppose, in the example above, the mass is pulled down a further 10 cm and released from rest. Does the string go slack?

Total extension is 15 cm . EPE stored is $\frac{1}{2} 0.8 \times 0.15^{2}=0.009 \mathrm{~J}$
GPE gained by mass in being raised 15 cm is

$$
0.004 \times 10 \times 0.15=0.006 \mathrm{~J}
$$

As $\mathrm{EPE}>\mathrm{GPE}$ the mass is still moving upwards when the string goes slack. Its speed is given by
$\frac{1}{2} \times 0.004 \times v^{2}=0.009-0.006$
so $v^{2}=1.5$; speed is $1.22 \mathrm{~m} \mathrm{~s}^{-1}$ (3 s.f.)
A light buffer of natural length 50 cm has a modulus of elasticity 100 kN .
(i) What force will compress it to half its length?
(ii) How much energy is stored in the buffer when it is compressed to 20 cm .
(i) $T=\frac{\lambda}{l} x=\frac{100000 \times 0.25}{0.5}=50 \mathrm{kN}$
(ii) $\mathrm{EPE}=\frac{1}{2 \times 0.5} 10^{5}(0.5-0.2)^{2}=9000 \mathrm{~J}$
E.g. A light spring of stiffness $50 \mathrm{~N} \mathrm{~m}^{-1}$ can project a 50 g stone 5 m vertically. How much compression in the spring is required?

$$
\begin{aligned}
\frac{1}{2} k x^{2} & =m g h \Rightarrow x=\sqrt{\frac{2 m g h}{k}} \\
& =\sqrt{\frac{2 \times 0.05 \times 9.8 \times 5}{50}}=0.313 \mathrm{~m}
\end{aligned}
$$

References:
Chapter 3
Pages 82-87

Example 3.1 Page 87

## References:

Chapter 3
Pages 89-95

Example 3.2
Page 89

Exercise 3B Q. 4

## Oscillating motion

- Suppose a particle oscillates about a central position, O , on the $x$-axis.
- The particle moves between two points, $x$ $=-a$ and $x=+a$. The distance $a$ is called the amplitude of the motion.
- The motion repeats itself in a cyclic fashion. The number of cycles per second is called the frequency, denoted by $v$.
- The motion repeats itself after time $T$. The time interval, $T$, is called the period: it is the time for a complete cycle of the motion.

The frequency is the reciprocal of the period.

$$
\text { So } v=\frac{1}{T}
$$

The S.I. unit for frequency is the Hertz. 1 Hertz is one cycle per second.

## Simple Harmonic Motion

The acceleration is proportional to the magnitude of the displacement from the centre point of the motion and is directed towards this centre point.
If $x$ is the displacement the SHM equation is
$\ddot{x}=-\omega^{2} x$ which gives $v^{2}=\omega^{2}\left(a^{2}-x^{2}\right)$
In addition:
$\ddot{x}$ is directed towards O , so the force causing the action must also be directed towards O .
$\ddot{x}$ (and therefore the force) $=0$ when $x=0$.
$|\ddot{x}|$ has a maximum value at the extremes $x= \pm a$.
The maximum velocity is when $x=0$ and is $\pm a \omega$.
The period, $T=\frac{2 \pi}{\omega}$.

## SHM as a function of time.

Any motion of the form $x=a \sin \omega t$ is a solution of the SHM equation since
$x=a \sin \omega t \Rightarrow v=\dot{x}=a \omega \cos \omega t$
$\Rightarrow \ddot{x}=-a \omega^{2} \sin \omega t=-\omega^{2} x$
and so is a solution of the SHM equation
The solution above is a particular solution of the more general form

$$
x=C \sin (\omega t+\varepsilon)
$$

(See next page for a list of expressions for SHM.)
Also, given $v=\dot{x}=a \omega \cos \omega t, v^{2}=a^{2} \omega^{2} \cos ^{2} \omega t$
$\Rightarrow v^{2}=a^{2}\left(1-\sin ^{2} \omega t\right)=a^{2}-x^{2}$
So $v^{2}=a^{2}-x^{2}$
E.g. The height, $h \mathrm{~m}$, of water in a harbour may be modelled over a short time by simple harmonic motion, with period 12 hours . At high water the depth of water is 6 m greater than the depth at low water.
$\omega=\frac{2 \pi}{T}=\frac{\pi}{6}$ rads per hour.
Amplitude of oscillation $=3 \mathrm{~m}$.
Maximum speed $=a \omega=\frac{\pi}{2}$ metres per hour.

$$
=\frac{100 \pi}{60 \times 2} \approx 2.62 \mathrm{~cm} \text { per minute (3.s.f.). }
$$

Assuming $h=0$ when $t=0, h=3 \sin \left(\frac{\pi}{6} t\right)$
Checking gives $\dot{h}=\frac{\pi}{2} \cos \left(\frac{\pi}{6} t\right), \ddot{h}=-\frac{\pi^{2}}{12} \sin \left(\frac{\pi}{6} t\right)$ $\Rightarrow \ddot{h}=-\left(\frac{\pi}{6}\right)^{2} h=-\omega^{2} h$ as required.
E.g. An oscillating weight on the end of a light spring completes 48 cycles per minute. The amplitude is estimated as 5 cm . If the oscillation begins at the equilibrium point, write an equation of motion and find the greatest speed.
Let $x$ be the displacement from the equilibrium point.
$v=48 \mathrm{cpm}=\frac{48}{60} \mathrm{~Hz} \Rightarrow T=\frac{60}{48}$
$\omega=\frac{2 \pi}{T}=\frac{2 \pi \times 48}{60}=\frac{8 \pi}{5}(=5.0265)$
$x=0.05 \sin \frac{2 \pi \times 48}{60} t=0.05 \sin 1.6 \pi t$
$\operatorname{Max} v=a \omega=0.251 \mathrm{~m} \mathrm{~s}^{-1}$ (3.s.f.)
Equation of motion: $\ddot{x}=-\omega^{2} x=-25.27 x$ (4 s.f.)
E.g. If $t=0$ at the middle of the motion then when $t=0, x=0$.
Using $x=C \sin (\omega t+\varepsilon)$ we have $0=a \sin \varepsilon \Rightarrow \varepsilon=0$
giving $x=a \sin \omega t$
If $t=0$ at one end of the motion then when $t=0, x= \pm a$.
Again, using $x=C \sin (\omega t+\varepsilon)$
Taking $x=a$ when $t=0 \Rightarrow a=a \sin \varepsilon \Rightarrow \varepsilon=\frac{\pi}{2}$
giving $x=a \sin \left(\omega t+\frac{\pi}{2}\right)=a \cos \omega t$
Taking $x=-a$ when $t=0 \Rightarrow x=-a \cos \omega t$

## References: <br> Chapter 3 <br> Pages 99-105

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Q. 2, 11

## References:

Chapter 3
Pages 107-109

Example 3.6
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Exercise 3C
Q. 8

References:
Chapter 3
Pages 115-119

Example 3.7
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Exercise 3D Q. 3

## References:

Chapter 3
Pages 119-124

Exercise 3D Q. 3

## Alternative forms of the solution of the SHM equation

Motion in a straight line of the form

$$
x=A \sin \omega t+B \cos \omega t
$$

or $\quad x=C \sin (\omega t+\varepsilon)$
or $\quad x=C \cos (\omega t+\varepsilon$ ')
satisfy the equation $\ddot{x}=-\omega^{2} x$
and the motion is SHM.

## Circular motion and SHM

If a particle is moving at constant angular speed $\omega$ round a circle of radius $a$ whose centre is O and the $x$-axis is a diameter, then the $x$ coordinate of its position is given by
$x=a \cos \theta$ where $\theta=\omega t$ and $\omega$ is the angular speed.
$\Rightarrow x=a \cos \omega t$ and $\dot{x}=-a \omega \sin \omega t$
So $\ddot{x}=-a \omega^{2} \cos \omega t=-\omega^{2} x$
Use $x=a \sin \omega t$ when $t=0$ at the centre of motion.
Use $x=a \cos \omega t$ when $t=0$ at the extreme of the motion in the positive direction.

## The simple pendulum

In tangential direction: $m l \ddot{\theta}=-m g \sin \theta$
For small $\theta, \sin \theta \approx \theta$ giving $\ddot{\theta} \approx-\frac{g \theta}{l}$
This equation of motion is the SHM equation. The solution has period $T=2 \pi \sqrt{\frac{l}{g}}$. The solution of the equation may be written as $\theta=\alpha \sin \left(\frac{g}{l} t+\varepsilon\right)$ where

$m g$ $\alpha$ is the maximum amplitude. Note that the equation for motion is the angular displacement of the pendulum.

## Oscillating Springs

If a particle is suspended from a perfectly elastic light spring of natural length $l_{0}$, and modulus of elasticity $\lambda$, then the equation of motion of oscillation is given by

$$
\ddot{x}=-\frac{\lambda}{m l_{0}} x
$$

where $x$ is the displacement from the equilibrium position. This is the standard equation for SHM with $\omega^{2}=\frac{\lambda}{m l_{0}} ;$ This gives a period of $2 \pi \sqrt{\frac{m l_{0}}{\lambda}}$

Consider the motion of a particle in a straight
line where the displacement, $x$, is given by
$x=2 \sin \left(\frac{\pi}{4} t\right)+3 \cos \left(\frac{\pi}{4} t\right)$.
$\dot{x}=\frac{\pi}{2} \cos \left(\frac{\pi}{4} t\right)-\frac{3 \pi}{4} \sin \left(\frac{\pi}{4} t\right)$
$\ddot{x}=-\frac{\pi^{2}}{8} \sin \left(\frac{\pi}{4} t\right)-\frac{3 \pi^{2}}{16} \cos \left(\frac{\pi}{4} t\right)$

$$
=-\frac{\pi^{2}}{16}\left(2 \sin \left(\frac{\pi}{4} t\right)+3 \cos \left(\frac{\pi}{4} t\right)\right)=-\frac{\pi^{2}}{16} x
$$

So the motion is SHM.
Also $x=2 \sin \left(\frac{\pi}{4} t\right)+3 \cos \left(\frac{\pi}{4} t\right)$
$\Rightarrow x=\lambda\left(\sin \left(\frac{\pi}{4} t\right) \cos \varepsilon+\cos \left(\frac{\pi}{4} t\right) \sin \varepsilon\right)$
where $\lambda \cos \varepsilon=2$ and $\lambda \sin \varepsilon=3$
$\Rightarrow \lambda^{2}=13$ and $\cos \varepsilon=\frac{2}{\sqrt{13}} \Rightarrow \varepsilon=0.983$ (3 s.f.)
$\Rightarrow x=\sqrt{13} \sin \left(\frac{\pi}{4} t+0.983\right)$ (3 s.f.)
E.g. A simple pendulum is in the form of a light rod of length 1.2 m with a heavy particle at the end. It is pulled to one side through an angle of $20^{\circ}$ from the vertical and then released from rest. Find
(i) the period,
(ii) the time to reach the lowest point,
(iii) the angular speed at the lowest point.
(Take $g=9.8 \mathrm{~m} \mathrm{~s}^{-2}$.)
(i) $\omega^{2}=\frac{g}{l}=8.166 \ldots \Rightarrow \omega=2.858 \ldots$
$\Rightarrow$ Period $=\frac{2 \pi}{\omega} \approx 2.20$ s (3 s.f.)
(ii) Motion is modelled by $\theta=\alpha \cos \omega t$
where $\alpha=\frac{20 \times \pi}{180} \approx 0.349 \ldots \mathrm{rad}$
At lowest point $\theta=0 \Rightarrow \cos \omega t=0 \Rightarrow t=\frac{\pi}{2 \omega} \approx 0.5496 \ldots$ sec.
(iii) Speed $=a \omega=0.3490 \times 2.85773=0.9975 \approx 1 \mathrm{rad} \mathrm{s}^{-1}$.
E.g. How long should be a simple pendulum for a 1 second "tick".
2 second period so $\frac{T^{2} g}{4 \pi^{2}}=l=\frac{4 g}{4 \pi^{2}} \approx 0.993 \mathrm{~m}$

## Mechanics 3

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References:
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Exercise 4B
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Exercise 4B

$$
\text { Q. } 8
$$

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Exercise 4C Q. 4

## Volume of revolution about the $\boldsymbol{x}$-axis

 If the curve $y=\mathrm{f}(x)$ between the ordinates $x=a$ and $x=b$ is rotated through $360^{\circ}$ about the $x$-axis then the volume of the resulting solid is given by:$$
V=\pi \int_{a}^{b} y^{2} \mathrm{~d} x
$$

Note that $y$ must be replaced by $\mathrm{f}(x)$ before integrating.

## Volume of revolution about the $\boldsymbol{y}$-axis

If the curve $x=\mathrm{f}(y)$ between $y=c$ and $y=d$ is rotated through $360^{\circ}$ about the $y$-axis then the volume of the resulting solid is given by:

$$
V=\pi \int_{c}^{d} x^{2} \mathrm{~d} y
$$

Note that $x$ must be replaced by $\mathrm{f}(y)$ before integrating.

Centres of mass (c.o.m)
For a volume of revolution about the $x$-axis:

$$
\bar{x} \int_{a}^{b} y^{2} \mathrm{~d} x=\int_{a}^{b} x y^{2} \mathrm{~d} x, \quad \bar{y}=0
$$

For a volume of revolution about the $y$-axis:

$$
\bar{y} \int_{c}^{d} x^{2} \mathrm{~d} y=\int_{c}^{d} y x^{2} \mathrm{~d} y, \quad \bar{x}=0
$$

## Centres of mass of plane regions

For the lamina which is the area between the curve $y=\mathrm{f}(x)$, the $x$-axis, $x=a$ and $x=b$.

$$
A\binom{\bar{x}}{\bar{y}}=\binom{\int_{a}^{b} x y \mathrm{~d} x}{\int_{a}^{b} \frac{y^{2}}{2} \mathrm{~d} y} \quad \text { where } A=\int_{a}^{b} y \mathrm{~d} x
$$

For the lamina which is the area between the curve $x=\mathrm{f}(y)$, the $y$-axis, $y=c$ and $y=d$.
$A\binom{\bar{x}}{\bar{y}}=\binom{\int_{c}^{d} \frac{x^{2}}{2} \mathrm{~d} y}{\int_{c}^{d} x y \mathrm{~d} y} \quad$ where $A=\int_{c}^{d} x \mathrm{~d} y$

## Mechanics 3

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Competence statements g 1, 2, 3, 4, 5
(C) MEI
E.g.1. The line $y=x+1$ from $x=0$ to $x=2$ is rotated through $360^{\circ}$ about the $x$-axis. Find the volume of the solid formed.

$$
\begin{aligned}
& V=\pi \int_{0}^{2} y^{2} \mathrm{~d} x=\pi \int_{0}^{2}(x+1)^{2} \mathrm{~d} x=\pi \int_{0}^{2}\left(x^{2}+2 x+1\right) \mathrm{d} x \\
& =\pi\left[\frac{x^{3}}{3}+x^{2}+x\right]_{0}^{2}=\pi\left(\frac{8}{3}+4+2\right)=\frac{26 \pi}{3}
\end{aligned}
$$

E.g.2. The curve $y=x^{2}-1$ from $y=1$ to $y=3$ is rotated through $360^{\circ}$ about the $y$-axis. Find the volume of the solid formed.

$$
\begin{aligned}
& V=\pi \int_{1}^{3} x^{2} \mathrm{~d} y=\pi \int_{1}^{3}(y+1) \mathrm{d} y \\
& =\pi\left[\frac{y^{2}}{2}+y\right]_{1}^{3}=\pi\left(\frac{9}{2}+3\right)-\pi\left(\frac{1}{2}+1\right)=6 \pi
\end{aligned}
$$

## Centres of mass of standard shapes

## Solidshape

## Volume Height of c.o.m.

 above plane baseSolid hemisphere, radius $r \quad \frac{2}{3} \pi r^{3} \quad \frac{3}{8} r$
Solid cone, height $h$, radius $r \frac{1}{3} \pi r^{2} h \quad \frac{1}{4} h$

## Hollow shape Curved Height of c.o.m. surface area above base

Hollow hemisphere, radius $r \quad 2 \pi r^{2} \quad \frac{1}{2} r$
Hollow cone, height $h$, radius $r$ rrl $\quad \frac{1}{3} h$
Composite bodies may be made by combining simpler shapes in some way.
E.g. The c.o.m. of a frustum which is the lower half of a solid cone, height $h$.
The height of the frustum is $h / 2$.
Let the mass of cone be 8 m .
Then mass of cone cut off is $m$ and mass of frustum is 7 m .



