MEI Mechanics 1

Motion

Section 1: The language of motion

Notes and Examples

These notes contain subsections on
- Definitions
- Notes on the textbook

Definitions

Displacement is the shortest route between two points. A distance and direction are needed. It is a vector quantity.

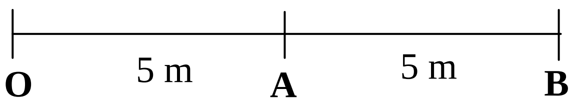
Position describes the location of something relative to a fixed point. This fixed point is usually referred to as the origin. Like displacement, a distance and direction are needed. It is a vector quantity.

The **distance** between two points involves no direction. It is just the physical distance between the chosen points. It is a scalar quantity.

The **distance travelled** between two points does not have to be the same as the distance between two points, described above. If you take a route which is not direct, them the distance travelled will be greater than the direct distance between the two points. Distance travelled is a scalar quantity.

To highlight the differences in the concepts of displacement, position, distance and distance travelled, look at Example 1 below.

Example 1



An object starts from A and travels to the right to B, then back through A to O and then back through A to stop at B.
Write down
(i)	The final displacement of the object
(ii)	The final position of the object
(iii)	The distance between its starting and finishing points
(iv)	The total distance travelled by the object

Solution
(i)	The final displacement from its initial position at A is 5 m to the right.
(ii)	Its final position is 10 m right of O.
(iii)	The distance between its starting and finishing points (A and B) is 5 m.

(iv) The total distance travelled is 25 m: 5 m from A to B, 10 m from B to O and another 10 m from O to B.

Notes on the textbook

Figure 1.4, page 3
It is a common mistake to confuse the position - time graph with the path of the marble through the air. It is not. The path of the marble is straight up and down. The position - time graph shows how the marble's position, relative to the origin (your hand, in this case) changes with time.

Question mark, page 3
The negative position means that the particle is below the level of your hand. The level of your hand has been taken as the origin in this example.

MEI Mechanics 1

Motion

Section 2: Speed and velocity

Notes and Examples

These notes contain subsections on

- **Definitions**
- **Notes on the textbook**

Definitions

Speed is a scalar quantity; it just has a particular size, e.g. 5 ms^{-1}. No direction is given or implied. Speeds are always positive.

Velocity is a vector quantity. It must have a size and a direction, e.g. 6 ms^{-1} upwards.

Velocities can be either positive or negative.

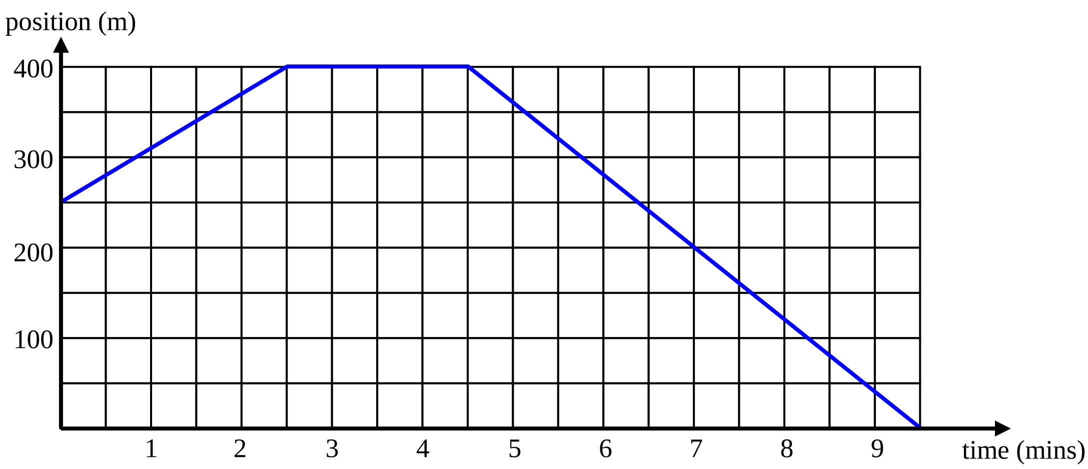
$$\textbf{Average speed} = \frac{\text{total distance travelled}}{\text{total time taken}}$$

$$\textbf{Average velocity} = \frac{\text{total displacement}}{\text{total time taken}}$$

From these formulae it is possible to see why average velocity can be negative if the resultant displacement is negative, whereas the total distance travelled will always be positive.

You can also find velocities as gradients of distance- time graphs (see p8).

Example 1
Mark gets off a bus, walks to the shop to buy some milk, and then walks home. The position-time graph below shows Mark's journey.



(i) What is Mark's final displacement?
(ii) What is the total distance Mark has walked?
(iii) In which part of the journey was Mark walking fastest?
(iv) How far is the bus stop from Mark's home?
(v) How far is the shop from the bus stop?
(vi) What is Mark's average speed for the whole journey?
(vii) What is Mark's average velocity for the whole journey?
(viii) What is Mark's average speed when he is actually walking?
(ix) Draw a speed-time graph for Mark's journey.
(x) Draw a velocity-time graph for Mark's journey.

Solution

(i) Mark's final displacement is -250 m.

The distance between his original position and final position. Note that it is negative because the direction of the shop has been chosen to be positive.

(ii) The total distance walked is 150 + 400 = 550 m.

(iii) Mark was walking fastest on the way home from the shop.

The last part of the graph is steeper than the first part.

(iv) The bus stop is 250 m from Mark's house.

(v) The shop is 150 m from the bus stop.

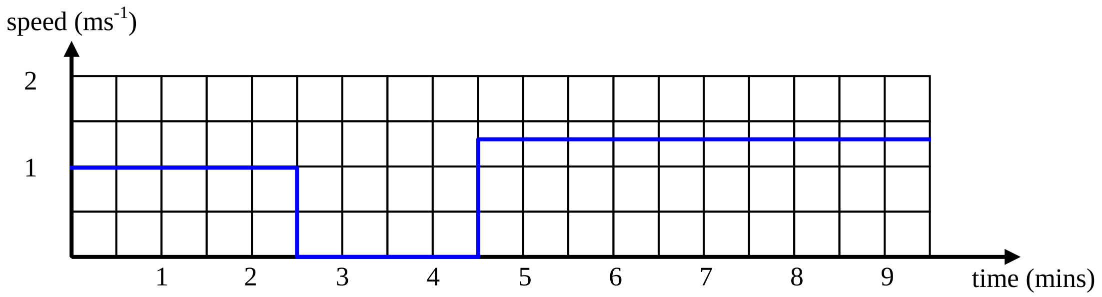
(vi) Total distance travelled = 550 m.
Total time taken = $9.5 \times 60 = 570$ seconds.
Average speed $= \dfrac{550}{570} = 0.965\,\text{ms}^{-1}$ (3 s.f.)

(vii) Total displacement = -250
Total time taken = $9.5 \times 60 = 570$ seconds.
Average velocity $= -\dfrac{250}{570} = -0.439\,\text{ms}^{-1}$ (3 s.f.)

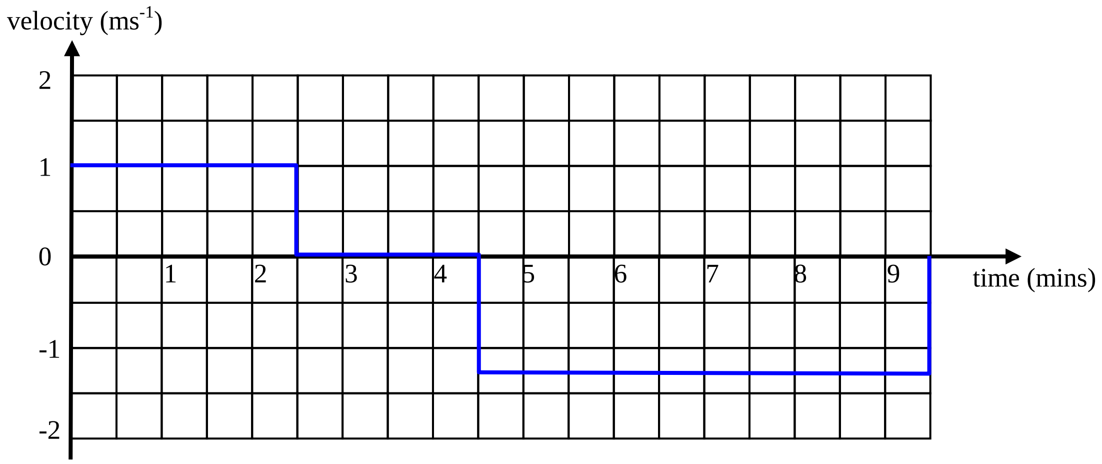
(viii) Total distance travelled = 550 m.
Total time taken when walking = $7.5 \times 60 = 450$ seconds.
Average speed $= \dfrac{550}{450} = 1.222\,\text{ms}^{-1}$ (3 s.f.)

(ix) Speed during journey to shop $= \dfrac{150}{2.5 \times 60} = 1 \text{ ms}^{-1}$.

Speed during journey from shop to home = $= \dfrac{400}{5 \times 60} = 1.33 \text{ ms}^{-1}$ (3 s.f.)



(x) Velocity during journey to shop = 1 ms^{-1}.
Velocity during journey from shop to home = -1.33 ms^{-1} (3 s.f.)



Notice in Example 1 that if you want to know how fast Mark walks, then the average speed and velocity calculations are not very helpful as they include the time he spent at the shop. The calculation of average speed when he was walking in part (vii) gives you the best idea of Mark's walking speed.

Notes on the textbook

Question mark, page 7
It is impossible to start and stop or jump to any new velocity instantaneously. More realistic graphs would have no sharp corners. However, Amy will be able to change velocity very quickly on her bicycle, so it would not make much difference to the answers if we tried to be more realistic. This is an example of mathematical modelling. It is easier to deal with the situation mathematically if we assume Amy can change velocity instantaneously and it will make little difference to our answers, so it is a reasonable modelling assumption to make.

Question mark, page 8

Velocity at H = 5 ms^{-1}
Velocity at A = 0 ms^{-1}
Velocity at B = -5 ms^{-1}
Velocity at C = -6 ms^{-1}
After the marble reaches the top, its velocity is negative so as the magnitude of the velocity (i.e. the speed) increases, the velocity decreases (remember $-2 < -1$).

MEI Mechanics 1

Motion

Section 3: Acceleration

Notes and Examples

These notes contain subsections on

- Acceleration
- Notes on the textbook

Acceleration

Acceleration is a measure of how much velocity is changing. This means it can affect both the speed and direction of motion. In this section you only look at motion along a straight line, so only two directions are possible, either forwards or backwards.

An acceleration of 2 ms^{-2} means that the velocity of a particle increases by 2 ms^{-1} every second (by 2 metres per second per second).
For example, if a car has an initial velocity of 6 ms^{-1} and an acceleration of 2 ms^{-2}, then after 1 second its velocity will be 8 ms^{-1}, after 2 seconds 10 ms^{-1} and after 3 seconds 12 ms^{-1} etc.

If a particle has a negative acceleration but a positive velocity, then it will slow down to a stop and then move in the opposite direction, with its speed steadily increasing.

Take care with the word deceleration. It is probably better not to use it. Use negative accelerations instead!

Take care that the units of acceleration are ms^{-2}. This is usually read as 'metres per second squared', or sometimes as 'metres per second per second'.

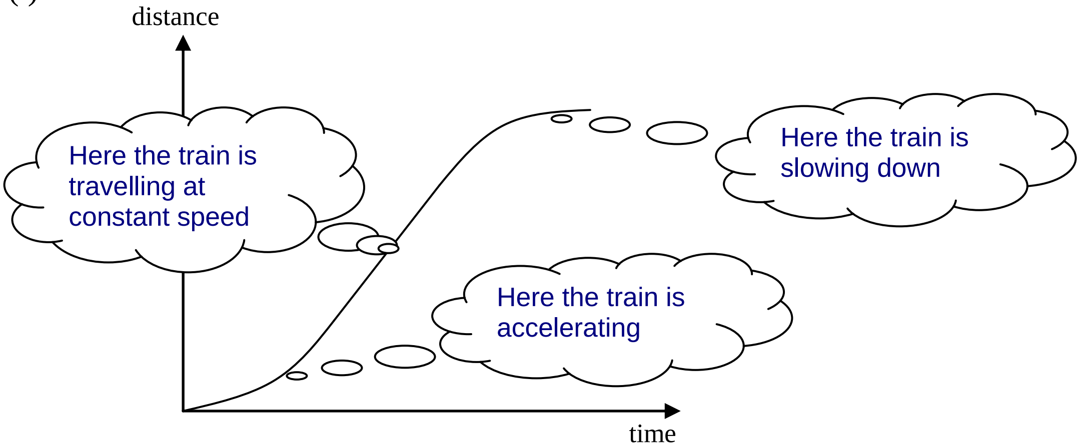
Accelerations can be found using the gradients of velocity – time graphs.

Example 1
When a local train leaves a station, it takes accelerates at a uniform rate of 3 ms^{-2} to its maximum speed of 60 ms^{-1}. It then maintains this speed for 2 minutes before slowing down uniformly to a halt at the next station. The whole journey takes 3 minutes.

(i) Sketch a distance-time graph for the journey.
(ii) Find the time the train takes to reach its maximum speed.
(iii) Draw the velocity-time graph for the journey.
(iv) What is the acceleration of the train in the last part of the journey?
(v) Draw the acceleration-time graph for the journey.

Solution

(i)



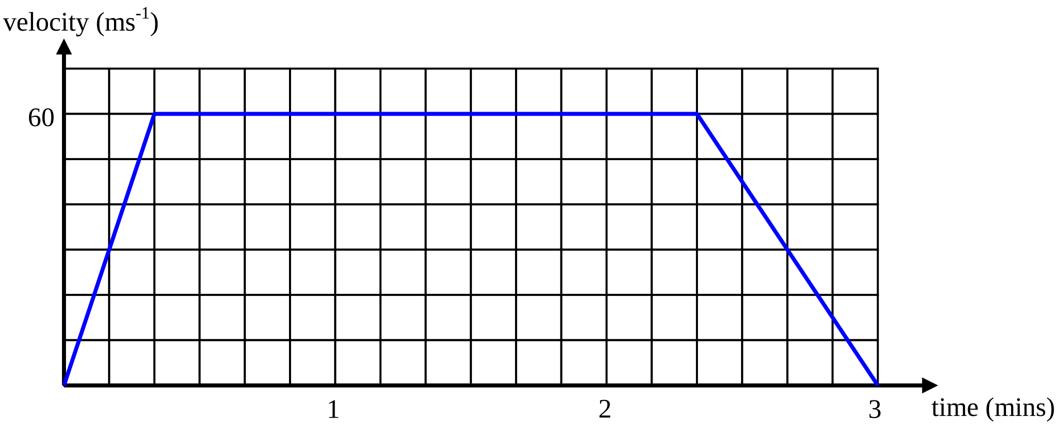
(ii) Acceleration $= \dfrac{\text{change in velocity}}{\text{time}}$

$$3 = \frac{60-0}{t}$$

$$t = \frac{60}{3} = 20$$

The train takes 20 seconds to reach its maximum speed.

(iii)

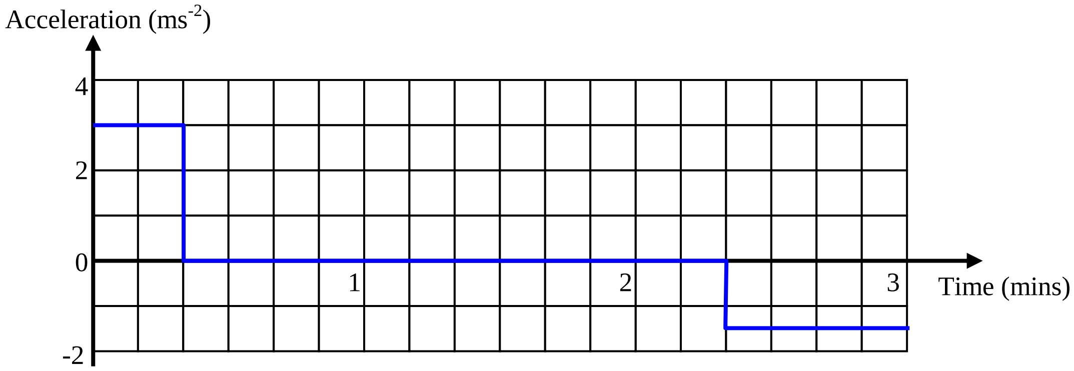


(iv) The last part of the journey takes 40 seconds, and the velocity changes from 60 ms^{-1} to 0.

$$\text{Acceleration } = \frac{\text{change in velocity}}{\text{time}}$$
$$= \frac{0-60}{40}$$
$$= -1.5$$

The acceleration is -1.5 ms^{-2}.

(v)



Notes on the textbook

Question mark, page 11

(i) Distance travelled can never decrease (although you can decrease displacement). Graph D is the only graph that never decreases. So the distance-time graph can only be D.

(ii) The velocity increases from zero to a maximum as the car accelerates away from the first set of traffic lights, then decreases from this maximum to zero as the car slows down to a stop for the second set of traffic lights. The velocity-time graph could be B, C or E as all these graphs start at zero, reach a maximum and then decrease to zero again.

(iii) As the car pulls away from the first set of traffic lights, its velocity is increasing so its acceleration must be positive. Once it starts to slow down for the second set of traffic lights its velocity is decreasing so its acceleration must be negative. Graph A is the only graph that moves from positive to negative in this way. So the acceleration-time graph must be A.

MEI Mechanics 1

Motion

Section 4: Using areas to find distances and displacements

Notes and Examples

These notes contain subsections on

- Distances and displacements from velocity-time graphs
- Notes on the textbook

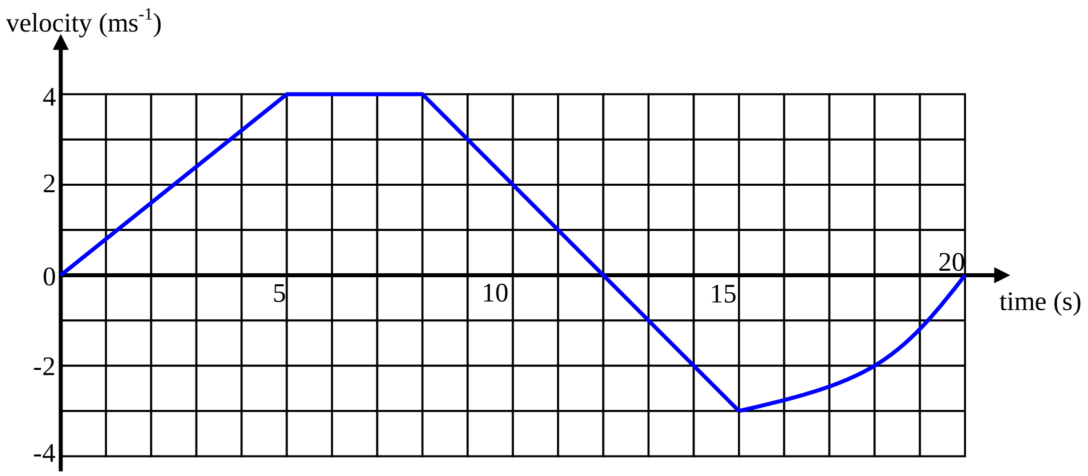
Distances and displacements from velocity-time graphs

The area under a velocity–time graph is usually equal to the distance a particle travels in the given time period, as long as the line does not cross the time axis (which is the same as saying the velocity is always positive). However, if the graph crosses the time axis (which is the same as saying the velocity becomes negative) as in example 1.2 on page 14, the situation changes so that the distance travelled is equal to the sum of the areas between the graph and the time axis, disregarding the negative sign, whereas the displacement is equal to the sum of the areas, incorporating the negative sign.

In example 1.2 on page 14, the distance travelled is 14 m, whereas the displacement is 10 m.

If the velocity-time graph is made up of a series of curves, then the area can be approximated by counting squares or by approximating the curve by straight lines to produce trapezia (as for the trapezium rule in Core 2).

Example 1
The diagram shows the velocity-time graph for the journey of a particle moving in a straight line.



(i) What is the acceleration of the particle during the first part of the journey?
(ii) How far does the particle travel in the first 12 seconds of its motion?
(iii) Estimate the distance travelled in the final 5 seconds of the motion.
(iv) What is the total distance travelled by the particle?
(v) What is the final displacement of the particle?
(vi) Find the average speed of the particle.
(vii) Find the average velocity of the particle.

Solution

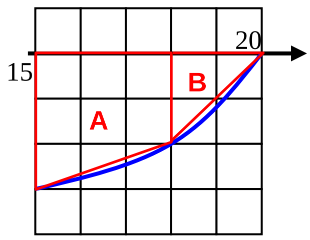
(i) During the first part of the journey the velocity of the particle increases from 0 to 4 ms^{-1} in 5 seconds.

$$\text{Acceleration} = \frac{\text{change in velocity}}{\text{time}} = \frac{4-0}{5} = 0.8$$

Acceleration = 0.8 ms^{-2}.

(ii) Distance travelled = area under graph between t = 0 and t = 12.
Area under graph between t = 0 and t = 5 is $\frac{1}{2}\times 5\times 4 = 10$
Area under graph between t = 5 and t = 8 is $3\times 4 = 12$
Area under graph between t = 8 and t = 12 is $\frac{1}{2}\times 4\times 4 = 8$
Total area = 10 + 12 + 8 = 30
Distance travelled = 30 m.

(iii) Splitting up the shape as shown in the diagram:
Area of trapezium A $= \frac{1}{2}(3+2)\times 3 = 7.5$
Area of triangle B $= \frac{1}{2}\times 2\times 2 = 2$
Total area = 9.5
Distance travelled ≈ 9.5 m.



(iv) Distance travelled between t = 12 and t = 15 is $\frac{1}{2}\times 3\times 3 = 4.5$
Total distance travelled = 30 + 4.5 + 9.5 = 44 m.

(v) Final displacement = 30 – (4.5 + 9.5) = 16 m.

(vi) $$\text{Average speed} = \frac{\text{total distance}}{\text{time}} = \frac{44}{20} = 2.2 \text{ ms}^{-1}$$

(vi) $$\text{Average velocity} = \frac{\text{total displacement}}{\text{time}} = \frac{16}{20} = 0.8 \text{ ms}^{-1}$$

MEI M1 Motion Section 4 Notes and Examples

Notes on the textbook

Question mark (near the top of page 15)
The distance travelled by the dog is given by the area under its speed-time graph. This can be estimated by approximating the area under the graph using trapezia.

In example 1.3, note that:

- When acceleration is uniform (constant) and positive, the speed - time graph for this section of the journey is a straight line with a positive gradient (equal to the acceleration).

- When the speed is constant (so that the acceleration is zero), the speed – time graph for this section of the journey is a horizontal straight line with gradient 0 (equal to the acceleration).

- When acceleration is uniform (constant) and negative, the speed – time graph for this section of the journey is a straight line with a negative gradient (equal to the acceleration).

Question mark (bottom of page 15)
If the journey is to take 2 minutes to cover 800 m and the maximum speed is maintained for 90 seconds, the trapezium ABCD must have area 800, the distance AD must be 120 m and the distance BC must be 90 m. From the formula for the area of a trapezium, this means that the value of v must always be the same.

The area of the trapezium is determined by AD, BC and v only. It is not affected by the gradients of AB or CD, so it does not matter how long the train takes to speed up or slow down.

MEI Mechanics 1

Modelling using constant acceleration

Section 1: The constant acceleration formulae

Notes and Examples

These notes contain subsections on

- **The constant acceleration formulae**
- **Applying the constant acceleration formulae**

The constant acceleration formulae

All the equations used in this section are derived from the two properties of the velocity - time graph that you have used in the previous chapter:

- The gradient of the graph gives you the acceleration,

  $$a = \frac{v-u}{t}$$

  which can be rearranged to give the more usual form

  $$v = u + at$$

- The area under the graph, gives you the displacement

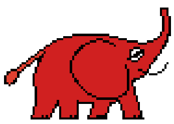
  $$s = \tfrac{1}{2}(u+v)t$$

In these equations

u = initial velocity
v = final velocity
a = acceleration
s = displacement
t = time

From the above two equations, three others can be derived, as seen on pages 23, 24.

These equations of motion or *suvat* equations are:



$$v = u + at$$
$$s = \tfrac{1}{2}(u+v)t$$
$$v^2 = u^2 + 2as$$
$$s = ut + \tfrac{1}{2}at^2$$
$$s = vt - \tfrac{1}{2}at^2$$

These equations are very important. **You should memorise them.**

Each equation includes 4 of the 5 variables, and each variable is missing from only one of the five equations.

Using the constant acceleration formulae

To do the questions it is usually best to start by writing out a list of the variables and filling in the ones that you know. In simple problems you will be given details of three of the variables have to find a fourth. The easiest way to solve such problems is to choose the correct *suvat* equation. This will be the equation that involves all of the variables that you are told in the question, together with the variable that you wish to calculate.

Example 1
A particle is accelerating at a constant 6ms^{-2}. After 8 seconds its displacement is 5m.
(i) What is its velocity after 8 seconds?
(ii) What was its initial velocity?
(iii) Describe the motion of the particle over these 8 seconds.

Solution
Using the information in the question you can write:

$s = 5$ m
$u =$?
$v =$?
$a = 6\text{ ms}^{-2}$
$t = 8$ s

(i) You know s, a and t and wish to know v. The *suvat* equation involving s, a, t and v is: $s = vt - \frac{1}{2}at^2$, so substituting in the appropriate values gives

$$\begin{aligned} 5 = 8v - \frac{1}{2}\times 6 \times 8^2 &\Rightarrow 5 = 8v - 192 \\ &\Rightarrow 197 = 8v \\ &\Rightarrow \frac{197}{8} = v \\ &\Rightarrow v = 24.6\text{ms}^{-1} \text{ (3s.f.)} \end{aligned}$$

Note: As a general rule, round your answers to 3 significant figures and state clearly that you have done so. Avoid rounding until your final answer, to avoid rounding errors.

(ii) You know *s*, *a* and *t* and wish to know *u*. The *suvat* equation involving *s*, *a*, *t* and *u* is: $s = ut + \frac{1}{2}at^2$, so substituting in the appropriate values gives

$$\begin{aligned} 5 = 8u + \tfrac{1}{2} \times 6 \times 8^2 &\Rightarrow 5 = 8u + 192 \\ &\Rightarrow -187 = 8u \\ &\Rightarrow \frac{-187}{8} = u \\ &\Rightarrow u = -23.4 \text{ ms}^{-1} \text{ (3s.f.)} \end{aligned}$$

(iii) Initially the particle has a speed of 23.4 ms^{-1} (3s.f.) and is moving in the negative direction. Acceleration is constant at 6 ms^{-1}. During the next 8 seconds the particle slows down until it is momentarily stationary and then speeds up to reach a speed of 24.6 ms^{-1} in the positive direction. At this time its displacement is 5 m in the positive direction from its starting point.

Important Note: Remember that the *suvat* equations can only be applied in situations where the acceleration is constant.

To practice using the constant acceleration formulae, you can use the interactive questions ***Choosing suvat equations*** and ***Solving suvat equations***.

MEI Mechanics 1

Modelling using constant acceleration

Section 2: Further examples

Notes and Examples

These notes contain subsections on
- Worked examples

Worked examples

Questions in this section usually involve a journey of two or more stages.
Write down the values of u, v, a, s and t for each stage.
Remember that the final velocity of one stage is the initial velocity of the next stage and that the accelerations across the two stages are often the same (though **NOT ALWAYS**)

Example 1
A cyclist reaches the top of a hill moving at 2 ms^{-1}, and accelerates uniformly so that during the sixth second after reaching the top he travels 13 m. Find his speed at the end of the sixth second.

Solution
The cyclist's displacement has increased by 13 m during the sixth second.

In the first 5 seconds	In the first 6 seconds
$u = 2$ ms^{-1}	$u = 2$ ms^{-1}
$v = ?$	$v = ?$
$a = a$ ms^{-2}	$a = a$ ms^{-2}
$s = x$ m	$s = (x + 13)$ m
$t = 5$ s	$t = 6$

13 m more than after 5 seconds

Using the equation which relates u, a, s and t, $s = ut + \frac{1}{2}at^2$

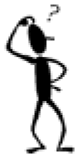
For $t = 5$: $x = 2 \times 5 + 12.5a$
For $t = 6$: $x + 13 = 2 \times 6 + 18a$

Solving these equations simultaneously gives $a = 2$ and $x = 35$.

To find the speed after 6 seconds:
$$v = u + at$$
$$= 2 + 2 \times 6$$
$$= 14$$
The speed at the end of the sixth second is 14 ms^{-1}.

MEI M1 Constant acc. Section 2 Notes and Examples

In Example 2, two methods of solution are given. Make sure you understand both.



Example 2
A train travels from rest at station A to stop at station B, a distance of 2100 m. For the first 20 seconds it accelerates steadily, reaching a speed of 25 ms^{-1}. It maintains this speed until the brakes are applied and the train brought to rest with uniform retardation over the last 125 m.
Find the retardation and the total time for the journey between the two stations.

Solution 1 (Using equations only)
Consider the three stages of the journey separately:

Stage 1 (accelerating)	Stage 2 (constant speed)	Stage 3 (decelerating)
$s = x$	$s = y$	$s = 125$
$u = 0$	$u = 25$	$u = 25$
$v = 25$	$v = 25$	$v = 0$
$a = a$	$a = 0$	$a = b$
$t = 20$	$t = t_1$	$t = t_2$

Also, from the total distance travelled we know that $x + y + 125 = 2100$.

Stage 1:
To find a use $v = u + at$: $25 = 0 + 20a$
$a = 1.25$

To find x use $s = \frac{1}{2}(u+v)t$, giving $x = \frac{1}{2}(0+25)\times 20 \Rightarrow x = 250$

Stage 2:
$250 + y + 125 = 2100 \Rightarrow y = 1725$ m

Using $s = ut + \frac{1}{2}at^2 \Rightarrow 1725 = 25t_1 + 0 \Rightarrow t_1 = \dfrac{1725}{25} = 69$

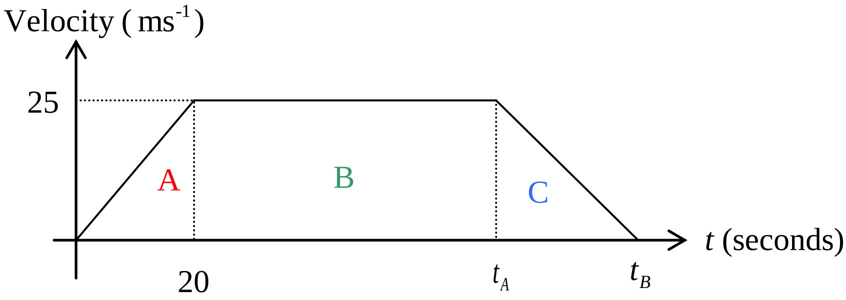
Stage 3:
To find b, use $v^2 = u^2 + 2as \Rightarrow 0 = 625 + 2\times 125b \Rightarrow b = \dfrac{-625}{250} = -2.5$

and to find t_2 use $s = \frac{1}{2}(u+v)t \Rightarrow 125 = \frac{1}{2}(25+0)t_2 \Rightarrow t_2 = \dfrac{250}{25} = 10$

(Use this equation rather than $v = u + at$ as it uses all given information rather than the quantities you have calculated – this is safer as you may have made errors in your calculations whereas given information must be assumed to be correct – this is a good general rule to follow for such questions)

The retardation is 2.5 ms^{-2}.
The total time for the journey is 20 + 69 + 10 = 99 seconds

Solution 2 (Using a velocity time graph)
Begin by sketching a graph and marking on it what you know:



Area A + B + C = 2100 (area under graph = distance travelled)
C = 125 (from question), A = 250 (using area of triangle), so:
B = 2100 – 125 – 250 = 1725.

So $(t_A - 20) \times 25 = 1725$ (using area of rectangle B)

$\Rightarrow t_A = 89$

So $\frac{1}{2}(t_B - 89) \times 25 = 125$ (Using area of triangle C)

$$\Rightarrow t_B = \frac{2 \times 125}{25} + 89 = 99$$

So total time between the stations is 99 seconds.

Acceleration = gradient of last line = $\frac{-25}{10} = -2.5 \text{ ms}^{-2}$, so retardation is 2.5 ms^{-2}.

MEI Mechanics 1

Forces and Newton's Laws of Motion

Section 1: Force diagrams and motion

Notes and Examples

These notes contain subsections on

- **Understanding Newton's Laws of Motion**
- **Types of force**
- **Force diagrams**

Understanding Newton's Laws of Motion

It will help you to understand and remember Newton's laws of Motion if you can relate them to your own experience. There are good examples in the textbook, but here are some more. Try discussing them with other people to help clarify your understanding.

Newton's first law: "Every object continues in a state of rest or uniform motion in a straight line unless acted on by a resultant external force."

1. Think about going quickly round a corner in a car. You feel that you are being thrown across the car, but this is not really what is happening. When you go around a corner, your velocity is changing because your direction of motion is changing. If you were not held in the car, by the friction between your body and the car seat, the car doors and your seat belt, you would continue to travel in a straight line, so the sensation of being thrown across the car is an illusion. What is actually happening is that the car is changing direction and if you are to change direction with it, a force must push you round the corner too. This force is provided by the friction between your body and the car seat, the car doors or your seat belt, which prevent you from continuing in a straight path and force you around the corner. The sensation of being thrown to one side is caused by being forced to change direction.

2. Think about doing an emergency stop in a car. You feel as though you are being thrown against the seatbelt (or through the windscreen if you are not wearing a seatbelt). What is actually happening is that your body will obey Newton's first law and continue to move forward with your original velocity, unless a force acts upon you body to slow it down. This force is provided by the seatbelt (or the windscreen). You are not really being thrown forward, you are being forced back.

3. Look at the necks of Formula 1 racing drivers. You will notice that they appear unusually thick in relation to most people. This is because racing cars turn corners extremely quickly and their drivers need to keep their heads as still as possible so that they can see clearly what is happening. To achieve this, their neck muscles must be very strong to force their

heads around the corners without them being thrown to one side. This is why racing drivers have thick necks!

You have probably heard of whiplash injuries, which can happen when a person's head is thrown backwards. They can occur when a car crashes into the back of a stationary car. People in the stationary car often suffer whiplash injuries, though these can be prevented if the car's seats are provided with head rests. Can you explain this using Newton's first law?

Newton's second law: "Resultant force = mass $\times$ acceleration or $\mathbf{F} = m\mathbf{a}$."

1. It takes a heavy lorry much more distance to stop than a car, even though its brakes can probably provide a greater braking force. This is because $\mathbf{a} = \frac{\mathbf{F}}{m}$ (from Newton's second law), so for a given force, the larger m is, the smaller **a** is, so a heavy lorry will slow down more slowly than a light car. Another similar example is oil tankers at sea. They can literally take miles to stop because they are so massive.

2. Try throwing a tennis ball as hard as you can, then try to throw a brick as far as you can. The distance something you throw (a projectile) will travel is dependent upon its initial velocity and the faster it is thrown (at a given angle), the further it will go (you will meet this in detail in chapter 6). With your maximum throwing force you will be able to throw the tennis ball at a greater initial speed than the brick because the brick is much heavier than the tennis ball, so the maximum force from your throwing arm will give it a smaller acceleration and hence a smaller initial speed when you let it go. The tennis ball will therefore go much further.
   (we have ignored air resistance and assumed the same angle of throw for both the brick and the tennis ball – both are reasonable assumptions)

Can you use Newton's second law to explain why rugby players tend to be big and heavy?

Newton's third law: "When one object exerts a force on another there is always a reaction which is equal and opposite in direction to the acting force."

1. If this were not the case, you would fall through the floor. When you are standing on the floor, the force of your weight acts vertically downwards. From Newton's first law, without a balancing force you would accelerate down through the floor. When you are standing stationary on the floor, the reaction force from the floor is exactly balancing your weight, so that the forces acting upon you are in equilibrium.

2. To feel a reaction force directly, try punching the wall! Boxers often break their hands by hitting their opponents.

MEI M1 Newton's Laws Section 1 Notes and Examples

There is a video (in three parts) at
http://openlearn.open.ac.uk/mod/oucontent/view.php?id=398660&section=1.1
which will help you to understand how Newton's ideas revolutionised thinking about force and motion.

Types of force

There are a number of types of force you need to consider and whose properties you need to be familiar with.

Weight	All particles have weight, unless they are defined as being 'light', in which case their weight does not effect the situation significantly and can be ignored. Weight is a force that always acts vertically downwards (towards the centre of the earth).
Resistance	This is a force that opposes the motion of a particle. Its direction is always opposite to the direction of motion. It can vary in size in some cases.
Reaction	When two objects come into contact with each other, each exerts a force on the other object. The direction of the force is perpendicular to the surface of contact.
Friction	This is a special type of resistance force. For a stationary object, any frictional force is always at exactly the correct size and direction to keep the object stationary. However, the frictional force has a maximum value. When the resultant force on an object exceeds this maximum, the object will move. The model of friction we use in Mechanics 1 assumes that whilst and object is moving, the frictional force is constant at this maximum value.
External forces	These are when objects are pushed or pulled. Driving forces and braking forces are examples of external forces.
Tension	Tension is a force that prevents two objects moving away from each other. It is often found in strings and rods. The cross-piece in a pair of step ladders is in tension as it prevents the two sides from separating. Tension forces 'pull'.
Thrust	Thrust is a force which prevents two objects coming together. It can be found in a rod but not in a string or rope. The legs of a chair provide a thrust force that prevents the seat of the chair falling to the floor. A rod between two objects can provide the thrust force required to keep them apart. A string or rope would just go slack. Thrust forces 'push'.

It is important to remember that strings and ropes can exert only tension forces, whereas rods can exert both tensions and thrusts.

Force Diagrams

These are the fundamental tools for making sense of mechanics questions. Unfortunately many students (usually unsuccessful ones) are reluctant to draw them. Try not to fall into this category! Keep them simple – use plain rectangles – no fancy artwork required! (See the example below)

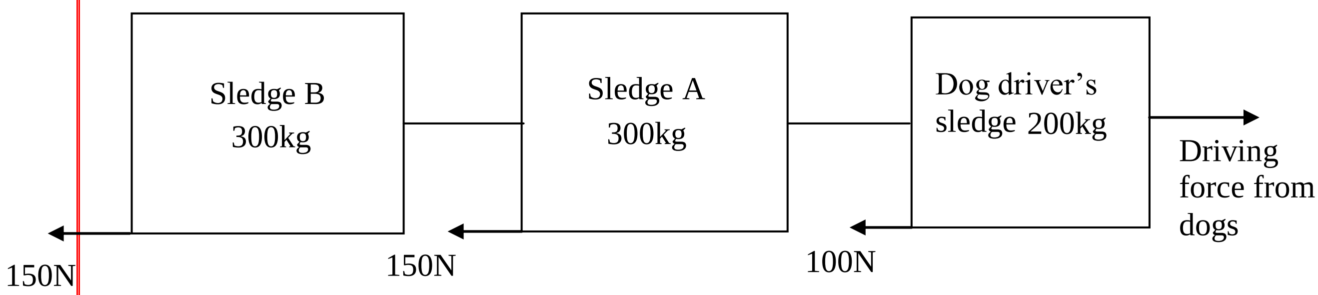
The idea of a force diagram is to show where forces are said to act, their line of action and, if possible, their direction.

A force diagram should be quite large (10cm) and show all forces acting. If necessary, do a number of diagrams showing the forces acting on the whole system and on different parts of the system separately.

The example below illustrates this idea.

Example 1

A team of husky dogs is pulling a 'sledge train' over the ice. The 'sledge train' consists of the dog driver's sledge, total mass (including the driver) 200kg and two trailer sledges strung behind it carrying supplies. Each of these trailer sledges has mass 300kg. The resistive forces experienced by each part of the sledge train are shown on the diagram below.



The sledge train is moving at constant velocity.

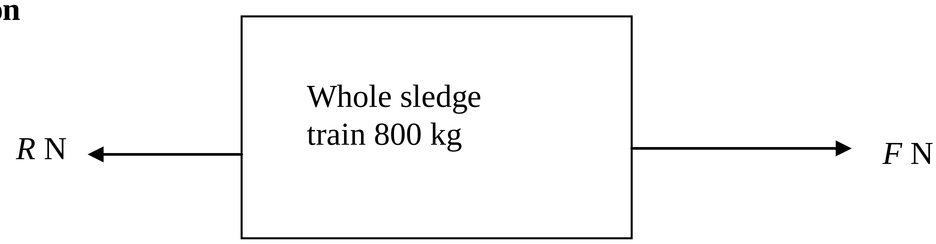
(a) By considering the forces on the system as a whole, calculate the driving force from the dogs.

(b) By considering the forces on the dog driver's sledge, calculate the tension in the coupling between the dog driver's sledge and sledge A.

(c) What is the tension in the coupling between sledge A and sledge B?

Solution

(a)

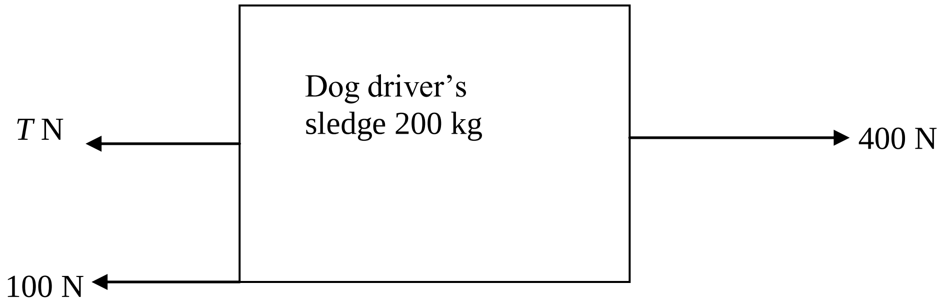


Since the sledge train is moving at constant velocity, the resultant force upon it must be 0, so $R = F$.

$R = 150 + 150 + 100 = 400$
(the combined total resistive force on the sledge train)

So the driving force from the dogs is $R = F = 400$ N.

(b)

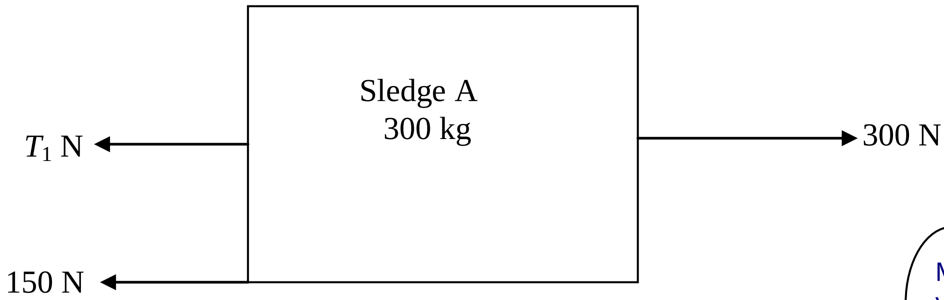


The whole sledge train is moving with constant velocity, so each part of it must be moving with constant velocity, so the forces on the dog driver's sledge must be in equilibrium.

$$T + 100 = 400 \Rightarrow T = 300$$

The tension in the coupling between the dog driver's sledge and sledge A is 300 N.

(c)



Sledge A must also be in equilibrium, so $T_1 + 150 = 300 \Rightarrow T_1 = 150$.

The tension in the coupling between sledges A and B is 150 N

Make sure you can get the same answer by considering the forces on sledge B

MEI Mechanics 1

Applying Newton's law along a line

Section 1: Introduction

Notes and Examples

These notes contain subsections on

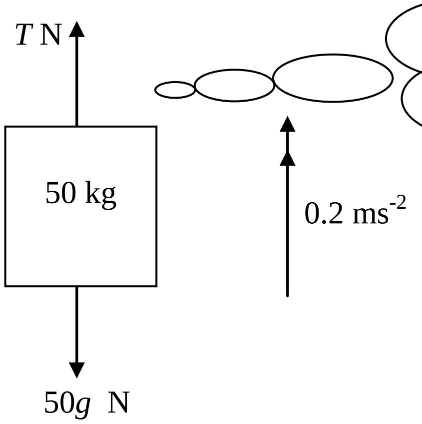
- Additional examples

Additional examples

Here are some additional examples, to give you some more ideas on how to approach these types of problems.

Example 1
A concrete block of mass 50 kg is lifted up the side of a building. The acceleration of the block is 0.2 ms^{-2}. Find the force in the rope.

Solution



The forces are
- the weight, $50g$ N, acting downwards
- the tension in the rope, T N, acting upwards

Resultant force = mass × acceleration

The block is moving up, so let up be positive.

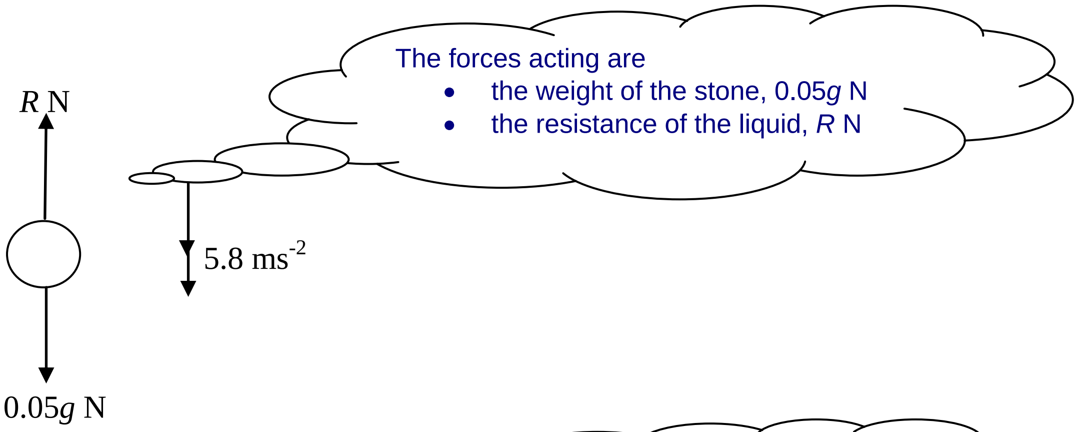
$$T - 50g = 50 \times 0.2$$
$$T = 490 + 10$$
$$T = 500$$

490 = 50 × 9.8; always use g = 9.8 unless instructed otherwise in the question

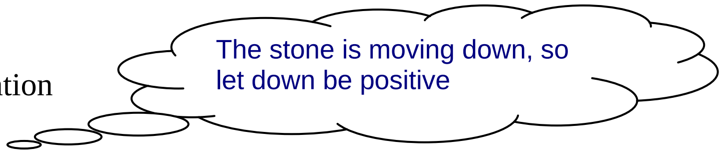
The tension in the rope is 500 N.

Example 2
A stone of mass 50 grams is dropped into some liquid and falls vertically through it with an acceleration of 5.8 ms^{-2}. Find the force of resistance acting on the stone.

Solution



Using resultant force = mass × acceleration



$$0.05g - R = 0.05 \times 5.8$$
$$0.49 - 0.29 = R$$
$$R = 0.2$$

So the resistance force acting on the stone is 0.2 N.

Example 3
A car of mass 700 kg is brought to rest in 7 seconds from a speed of 20 ms^{-1}. What constant force is necessary to produce this retardation?

Solution
The only force acting is the retardation.
The relationship:

Resultant force = mass × acceleration

cannot be used immediately as there are two unknowns, but there is sufficient information to calculate the acceleration, using the equations of motion.

$u = 20 \text{ ms}^{-1}$
$v = 0 \text{ ms}^{-1}$
$a = ?$
$s = ?$
$t = 7 \text{ s}$

Using $v = u + at$
$$0 = 20 + 7a$$
$$a = \frac{-20}{7} \text{ ms}^{-2}$$

So, using Resultant force = mass × acceleration:

$$F = 700 \times \frac{20}{7}$$
$$F = 2000$$

So the retarding force is 2000 N.

MEI Mechanics 1

Applying Newton's laws along a line

Section 2: Connected objects

Notes and Examples

These notes contain subsections on
- Solving problems involving connected particles
- Further examples

Solving problems involving connected particles

For a problem such as a car towing a caravan, note that there are three possible equations that you can write down: one for each object separately and one for the system as a whole. However, these three equations are not independent: each can be obtained by combining the other two in some way. So you cannot solve a problem with three unknowns, only ones with two unknowns. It is often easier to use the one for the whole system first – this will usually mean that you can avoid using simultaneous equations.

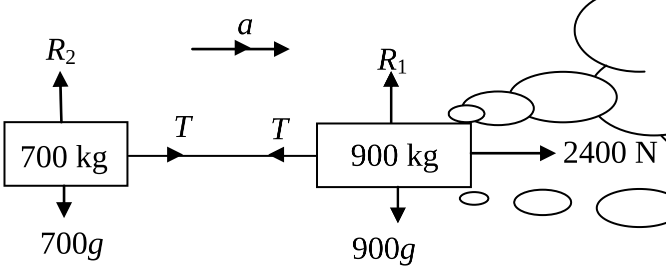
However, when you are dealing with problems involving pulleys, you cannot write down an equation of motion for the whole system, since the use of a pulley means that the particles are moving in different directions. You can only consider the system as a whole when the connected particles are moving in the same line, such as for one vehicle towing another.

Try the Flash resource ***Car pulling a caravan***. You can vary the masses and resistance forces, and then look at the force diagrams and calculate the acceleration.

Further examples

Example 1
A car of mass 900 kg tows a caravan of mass 700 kg along a level road. The engine exerts a forward force of 2400 N and there is no resistance to motion. Find the acceleration produced and the tension in the tow bar.

Solution



Draw a diagram and put on all forces, even those you may not need, e.g. reaction and weight. Include an acceleration arrow, pointing in the direction of motion of the system.

Notice that the tensions cancel one another out when you look as the system as a whole

For the whole system, in the direction of motion:

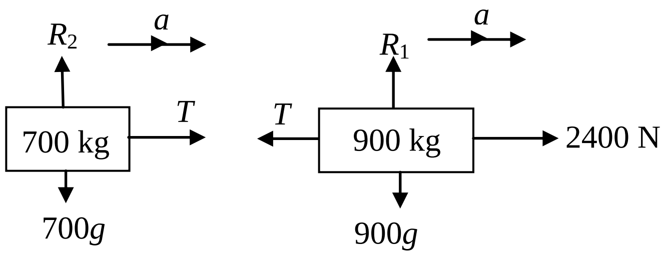
$F = ma$

$2400 = (700 + 900)a$

$a = 1.5$

So the acceleration is 1.5 ms^{-2}.

To find the tension in the tow bar, consider the forces on either the car or the caravan. It will work with either. The acceleration of both must be 1.5 ms^{-2}, because they are part of the same connected system.



Consider the forces in the direction of motion:

If you look at the caravan:

$T = 700 \times 1.5$

$T = 1050$

If you look at the car:

$2400 - T = 900 \times 1.5$

$2400 - 1350 = T$

$1050 = T$

So the tension in the tow bar is 1050 N.

Example 2

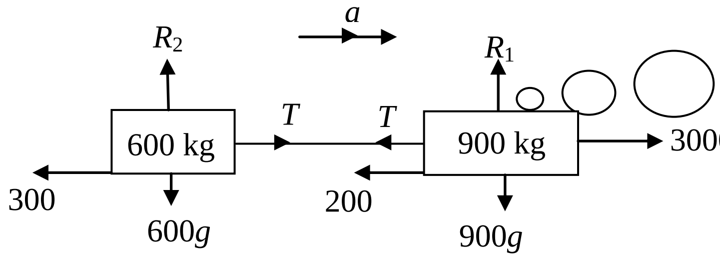
A car of mass 900 kg tows a trailer of mass 600 kg by means of a rigid tow bar. The car experiences a resistance of 200 N and the trailer a resistance of 300 N.

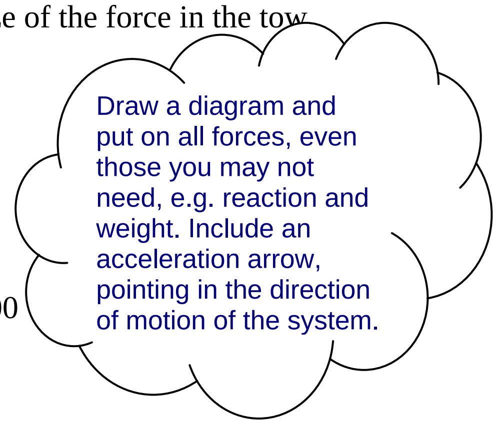
(a) If the car engine exerts a driving force of 3000 N, find the acceleration of the system and the tension in the tow bar.

(b) If the engine is switched off and the brakes are applied to the car, giving a retardation force of 500 N, what will be the retardation of the car, assuming other resistances are unchanged, and what is the nature and size of the force in the tow bar?

Solution

(a)



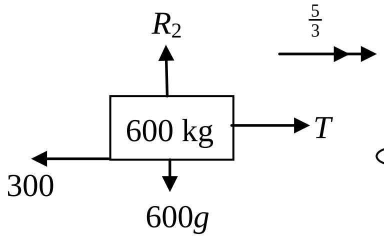


For the whole system:

$$3000 - 300 - 200 = (900 + 600) \times a$$
$$2500 = 1500a$$
$$\frac{2500}{1500} = a$$
$$\tfrac{5}{3} = a$$

The acceleration is $\frac{5}{3}$ ms^{-2}.

> To find the tension in the tow bar, look at the forces on the trailer. It is slightly easier than looking at the forces on the car, as there is one less force to consider.

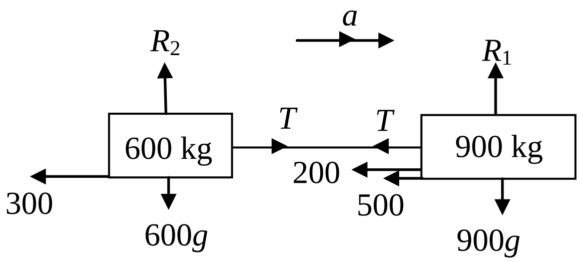


$$T - 300 = 600 \times \tfrac{5}{3}$$
$$T = 1300$$

So the tension in the coupling is 1300 N.

> Notice that there is now no driving force as the engine has been switched off.

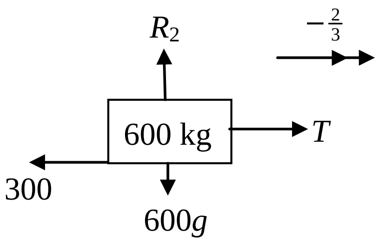
(b)



$$0 - (500 + 200 + 300) = (900 + 600) \times a$$
$$-1000 = 1500a$$
$$-\tfrac{2}{3} = a$$

So the acceleration is $-\frac{2}{3}$ ms^{-1}.

Again, looking at the forces only acting on the trailer:



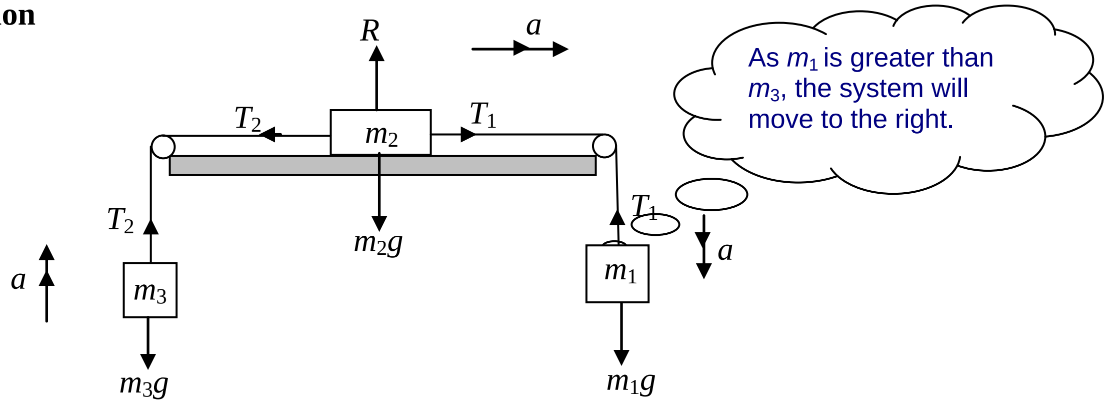
$$T - 300 = 600 \times -\tfrac{2}{3}$$
$$T = -100$$

So the tension in the tow bar is 100 N and as it is negative it must be keeping the car and trailer apart, so the tow bar is in compression or thrust.

Example 3
The diagram shows masses of m_1, m_2 and m_3 connected by a light inextensible string. Masses m_1 and m_3 hang vertically with $m_1 > m_3$. Mass m_2 lies on a smooth surface.

(i) Write down equations of motion for each of the masses.
(ii) In the case $m_1 = 8$ kg, $m_2 = 10$ kg and $m_3 = 5$ kg, find the acceleration of the system.

Solution
(i)



Mass m_1 moves down, so the equation of motion for m_1 is

$$m_1g - T_1 = m_1 a$$

Mass m_3 moves up, so the equation of motion for m_3 is

$$T_2 - m_3g = m_3a$$

Mass m_2 moves to the right so T_1 must be greater than T_2 so the equation of motion for m_2 is

$$T_1 - T_2 = m_2a$$

(ii) $m_1 = 8$, $m_2 = 10$, $m_3 = 5$.

The equations are:
$$8g - T_1 = 8a \quad \Rightarrow T_1 = 8g - 8a$$
$$T_2 - 5g = 5a \quad \Rightarrow T_2 = 5g + 5a$$
$$T_1 - T_2 = 10a$$

Substituting the first two equations into the third:
$$8g - 8a - (5g + 5a) = 10a$$
$$3g = 23a$$
$$a = \frac{3\times 9.8}{23} = 1.28 \text{ (3 s.f.)}$$

The acceleration of the system is 1.28 ms^{-2} (3 s.f.)

MEI Mechanics 1

Vectors

Section 1: Introduction

Notes and Examples

These notes contain subsections on

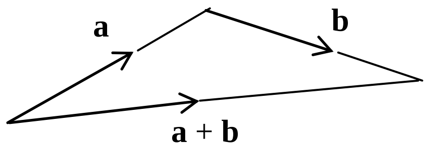
- **Adding vectors**
- **The components of a vector**
- **The magnitude and direction of a vector**
- **Unit vectors**

Adding vectors

Whenever you are doing work with vectors, draw a diagram. Sometimes it can be useful to use square paper. Although scale drawings are not acceptable for this level of work, an accurate diagram can help you check your answer.

As is so often the case in mathematics, clear notation is vital in vector work. Make sure you use the notations specified on page 79.

One way of adding vectors is to draw them ‘nose to tail’, i.e. where one finishes, the next one starts.



When you add vectors, you are finding a single vector that can replace two or more vectors.

The components of a vector

Finding the components of a vector is the reverse process of combining two vectors. It is often more convenient to split a vector into two perpendicular components, than to use the original vector itself; e.g. the weight of a block standing on an inclined (sloping) plane (surface) acts vertically down, but it is more useful to think of part of its weight acting down the slope, and another part acting perpendicular to the slope. You will use this idea extensively in chapter 7.

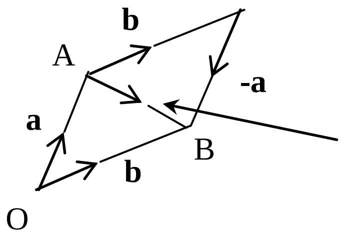
Sometimes you are given the components of a vector, either using the **i, j** notation, e.g. $4\mathbf{i}+\mathbf{j}$, or in column vector form, e.g. $\begin{pmatrix}4\\1\end{pmatrix}$

This is studied in more depth in the next unit and in chapter 7.

MEI M1 Vectors Section 1 Notes and Examples

Discussion point, page 87

This is an important result. The diagram below should help to illustrate it:



$\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$

The magnitude and direction of a vector

The magnitude of a vector is just its length and can be found using Pythagoras' Theorem. The direction of a vector needs more care and **a diagram will always help**. The angle is usually measured anticlockwise from the *x* axis (the **i** direction).

Make sure that you read Example 5.5 carefully.

Unit vectors

If you wanted to find a vector of magnitude 40, parallel to the vector $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$, the easiest way to do this is to find a unit vector parallel to $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$, and then multiply that vector by 40 to get the vector you require.

$\left|\begin{pmatrix} 4 \\ 1 \end{pmatrix}\right| = \sqrt{4^2 + 1^2} = \sqrt{17}$, so a unit vector parallel to $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ is $\frac{1}{\sqrt{17}}\begin{pmatrix} 4 \\ 1 \end{pmatrix}$, so a vector of magnitude 40, parallel to $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ is $\frac{40}{\sqrt{17}}\begin{pmatrix} 4 \\ 1 \end{pmatrix}$. Note that it is best to leave the square root in your answer unless you are required to give your answer to a specified degree of accuracy because if you use your calculator to work out the components of the vector as a decimal, you will have to round your answer. With the square root sign left in, it is perfectly accurate.

Note $\frac{40}{\sqrt{17}}\begin{pmatrix} 4 \\ 1 \end{pmatrix} = \frac{160}{\sqrt{17}}\mathbf{i} + \frac{40}{\sqrt{17}}\mathbf{j}$.

Vectors in three dimensions (page 91)

As a challenge (not too difficult!), prove that the magnitude of a general three dimensional vector, $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$, is $\sqrt{a^2+b^2+c^2}$

MEI Mechanics 1

Vectors

Section 2: Resolving vectors

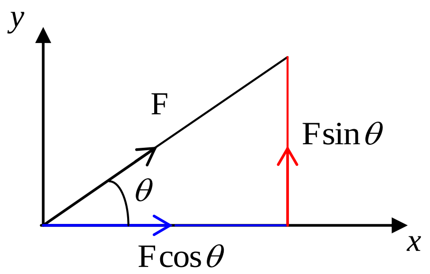
Notes and Examples

These notes contain subsections on
- **Resolving vectors**
- **Velocity triangles**

Resolving vectors

Resolving vectors is absolutely fundamental to mechanics. To start with you may find it useful to follow the method below. Once you become fluent (and you need to keep practising until you feel fluent), you will probably be able to resolve vectors without really thinking about it.

- Draw the vector, magnitude F
- Mark on your *x* and *y* axes
- Mark on one angle, which you know
- Make your vector into a right–angled triangle.
- Imagine swinging the vector through the angle until it lies on one of the other sides of the triangle
- This component is F$\times\cos\theta$
- The other component is F$\times\sin\theta$



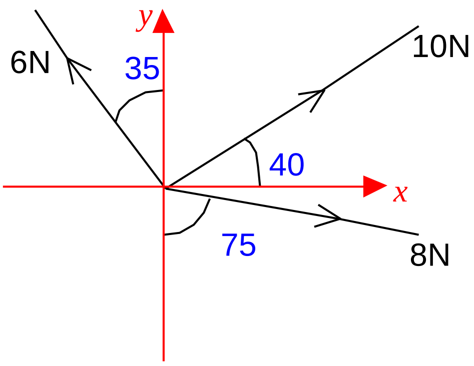
So in component form, $\mathbf{F} = \mathrm{F}\cos\theta\mathbf{i} + \mathrm{F}\sin\theta\mathbf{j}$ or $\begin{pmatrix}\mathrm{F}\cos\theta \\ \mathrm{F}\sin\theta\end{pmatrix}$.

Sometimes vectors are resolved parallel and perpendicular to a slope along which a particle is moving. The same principles apply; the *x* axis would be drawn parallel to the slope and the *y* axis perpendicular to the slope.

MEI M1 Vectors Section 2 Notes and Examples

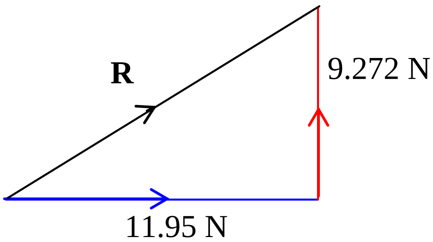
If you must resolve a number of vectors and find their resultant, it is best if you draw up a table.

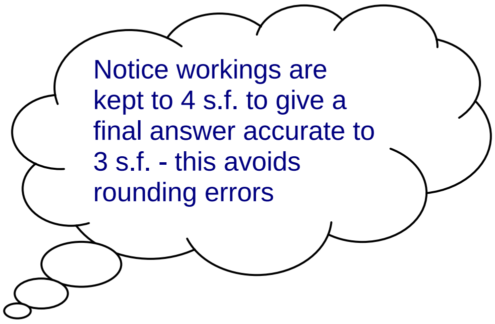
Example 1
What single force can replace the following system of forces?



Solution

Force	x direction	y direction
10 N	10cos 40 = 7.660	10 sin 40 = 6.428
8 N	8sin 75 = 7.727	-8cos 75 = -2.071
6 N	-6sin 35 = -3.441	6cos 35 = 4.915
Resultant force, R	11.95	9.272





The single force that can replace the system has magnitude:

$$|\mathbf{R}| = \sqrt{11.95^2 + 9.272^2} = 15.1 \text{ N (3s.f.)}$$

and direction:

$\tan^{-1}\left(\dfrac{9.272}{11.95}\right) = 37.8°$ (3 s.f.) anticlockwise from the positive x direction.

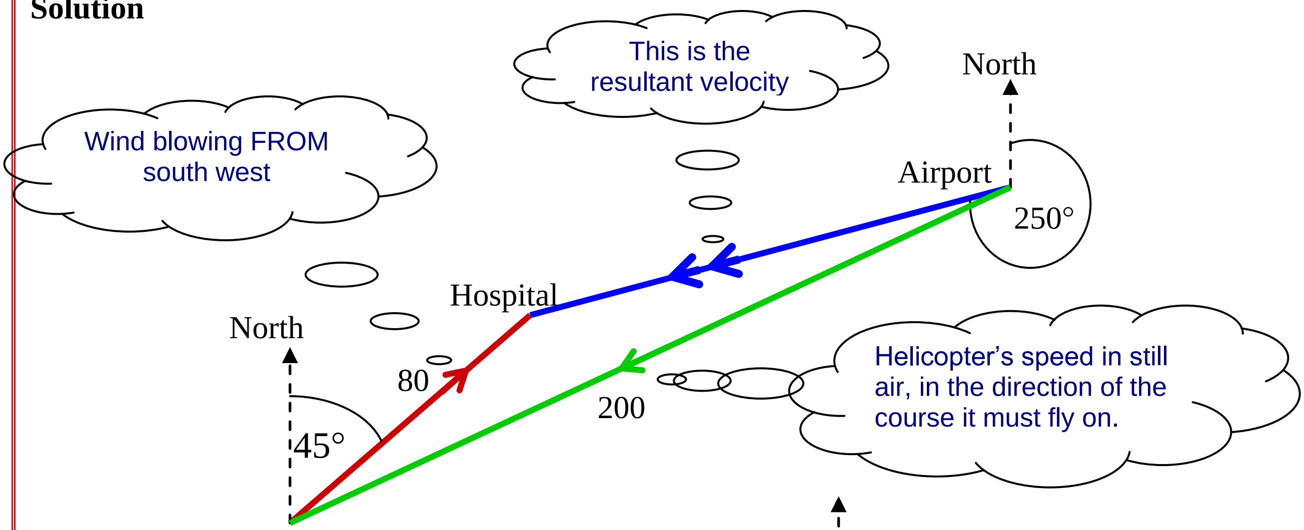
Vector triangles

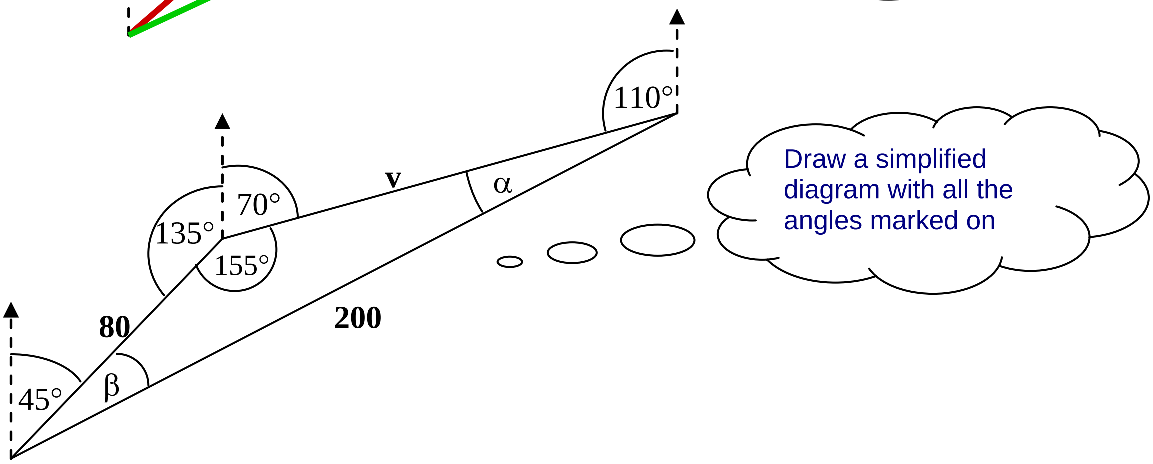
The work on relative velocity and vector triangles is just an application of the work above. In these problems, remember that the path between the two

places is the resultant path and it is usually made up of two parts. The sine and cosine rule are often useful in this work.
Here is a further example:

Example 2
A helicopter can fly at 200 kmh^{-1} in still air. It has to travel from an airport to a hospital. The hospital is 600 km from the airport on a bearing of 250°. The wind speed is 80 kmh^{-1} and it is blowing from the south west. Work out the course that the helicopter must fly on and the time that it takes to get to the hospital.

Solution





Using the sine rule: $\dfrac{\sin\alpha}{80} = \dfrac{\sin 155}{200}$

$$\sin\alpha = \frac{80\sin 155}{200}$$

$$\alpha = 9.7° \text{ (1 d.p.)}$$

The bearing on which the helicopter must fly is $360 - 110 - \alpha = 240°$ (to nearest degree).

$\beta = 180 - 155 - \alpha = 15.3°$

Using the sine rule again: $\dfrac{v}{\sin\beta} = \dfrac{200}{\sin 155}$

Alternatively you could use the cosine rule

$$v = \frac{200\sin 15.3}{\sin 155} = 124.9$$

The time taken $= \dfrac{600}{124.9} = 4.8$

The time taken is 4 hours 48 minutes (to nearest minute)

You may also like to look at this animated ***PowerPoint presentation*** showing the example above.

MEI Mechanics 1

Projectiles

Section 1: Introduction

Notes and Examples

These notes contain subsections on

- **Modelling assumptions**
- **General strategy for projectile questions**
- **Components of the velocity**
- **Finding the time of flight, range and maximum height**

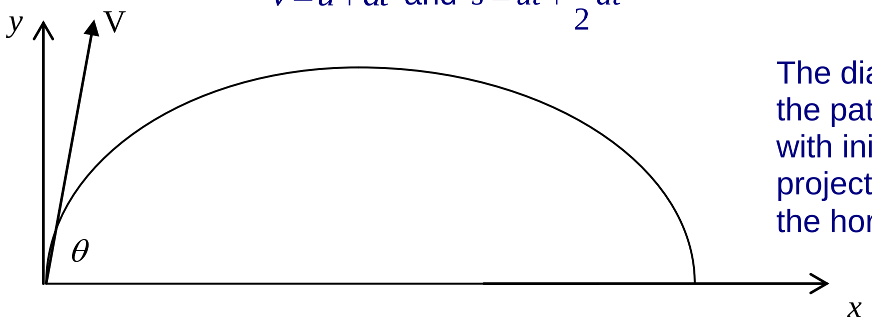
Modelling assumptions

The modelling assumptions at the foot of page 101 are very important. Without them, analysing projectiles would be much harder. In many situations these assumptions will not make a significant difference to the final answer, so they are reasonable. However, throwing a flat sheet of paper, for example, could not usefully be analysed without taking account of the effects of air resistance.

General strategy for projectile questions

The notes on page 102 show how the velocity is split into two components, horizontal and vertical. This is the standard way to solve projectiles questions.
The equations of motion are then applied to each component of velocity. The main ones used are:-

$$v = u + at \quad \text{and} \quad s = ut + \frac{1}{2}at^2$$



The diagram shows the path of a projectile with initial velocity V, projected at angle θ to the horizontal.

Vertically

$a_y = -g$

$u_y = V\sin\theta$

vertical velocity (using $v = u + at$)

$v_y = V\sin\theta - gt$

vertical displacement (using $s = ut + \frac{1}{2}at^2$)

$y = Vt\sin\theta - \frac{1}{2}gt^2$

Horizontally

$a_x = 0$

$u_x = V\cos\theta$

horizontal velocity (using $v = u + at$)

$v_x = V\cos\theta$

horizontal displacement (using $s = ut + \frac{1}{2}at^2$)

$x = Vt\cos\theta$

where t is the time of flight.

MEI M1 Projectiles Section 1 Notes and Examples

Components of the velocity

The vertical component

When the vertical component of velocity is positive, the particle is rising.

- At the instant the vertical component of velocity is 0, the particle is at maximum height.
- When the vertical component of velocity is negative, the particle is falling.

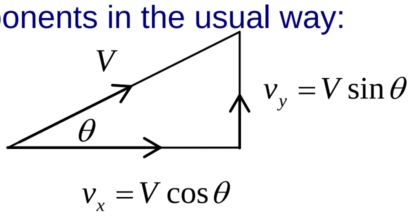
The horizontal component

Remember: THE HORIZONTAL VELOCITY REMAINS CONSTANT.

- It is worth noting that a particle which is fired from the top of a cliff at 100 ms^{-1} horizontally and another which is dropped from the top of the same cliff at the same time will land on the ground at the same time! This is because they both have an initial vertical component of velocity of 0 ms^{-1}. Their initial horizontal components of velocity will have no effect on their motion in the vertical direction.

Direction of flight

- The direction of flight depends upon the ratio of the horizontal and vertical velocities.
- As the horizontal velocity remains constant, the direction of flight changes because of the change in the vertical velocity.
- The direction of flight can be obtained by combining the velocity components in the usual way:



$$\tan\theta = \frac{v_y}{v_x}$$

Finding the time of flight, range and maximum height

Time of flight

The time of flight can be found in two ways:-

- Use $v = 0$ in $v = u + at$ to find the time to maximum height, and then double it. *This only works if the starting and finishing points are on the same level.*
- Put $y = h$ in $s = ut + \frac{1}{2}at^2$, where h is the vertical displacement from its starting point when the particle lands, then solve this quadratic to give t. If the projectile starts and stops at the same level, $h = 0$. There will be two solutions to the quadratic, but the time when the particle lands must be the greater *(think about why this is the case).*

MEI M1 Projectiles Section 1 Notes and Examples

Range

The range is found by multiplying the time of flight with the horizontal component of the velocity.
(*Remember, for a projectile, the horizontal component of velocity is* **CONSTANT**).
It may be stating the obvious, but the range is always increasing whilst the particle is off the ground.

Maximum height

At the maximum height, the vertical component of velocity is 0, so use $v=0$ in $v^2=u^2+2as$ to get the maximum height and in $v=u+at$ to get the time to maximum height.

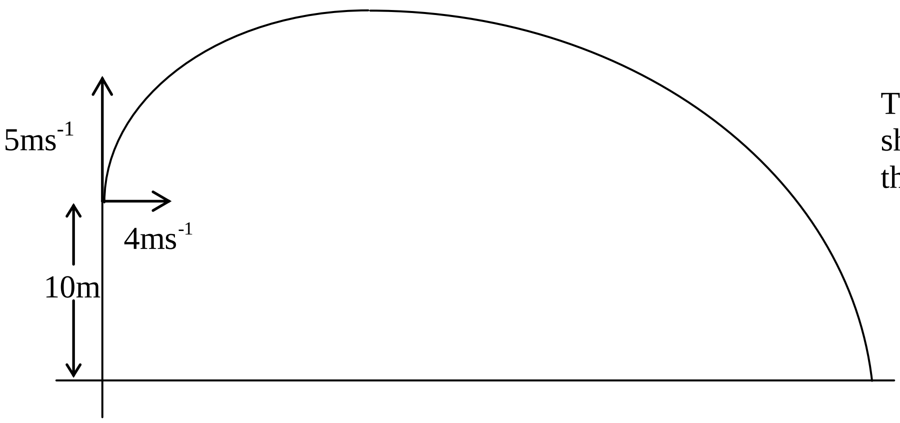
The ***Component interactive spreadsheet*** allows you to investigate the flight of a projectile as the vertical and horizontal components of the velocity vary.

The ***Angle interactive spreadsheet*** allows you to investigate how varying the angle of projection affects the range of the projectile.

Example 1

A ball is projected with a velocity $\begin{pmatrix}4\\5\end{pmatrix}$ from a position $\begin{pmatrix}0\\10\end{pmatrix}$. Assuming g = 10ms^{-2}

find:
(i) the maximum height
(ii) the time of flight
(iii) the range.
(iv) the angle of flight after 1 second.



The curved line shows the path of the projectile

Solution

(i) At the maximum height, the vertical velocity is 0.
So, using the equation

$$v^2=u^2+2as$$
$$0^2=5^2+2(-10)s$$
$$s=\frac{25}{20}=1.25$$

So the maximum height is 1.25m above the starting point, 11.25m above the ground.

(ii) The time of flight can be obtained using $s = ut + \frac{1}{2}at^2$, considering the vertical motion of the projectile. Here *s* will be –10 m, as we want the time when the particle hits the ground. At this time the particle is 10 m below the starting point.

$$s = ut + \tfrac{1}{2}t^2$$
$$-10 = 5t + \tfrac{1}{2}(-10)t^2$$
$$5t^2 - 5t - 10 = 0$$
$$5(t+1)(t-2) = 0$$

So $t = -1$, which is impossible, or $t = 2$.

The time of flight is 2 seconds.

(iii) The range is just the horizontal velocity, which remains constant, multiplied by the time of flight.

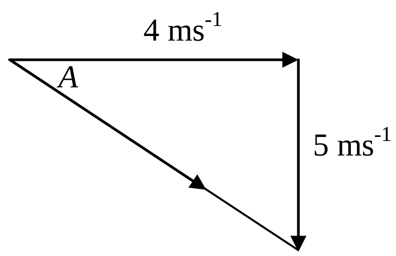
So the range is $4 \times 2 = 8$ m

(iv) The direction of the flight is given by the ratio of the velocities.

After 1 second, the vertical velocity is found using $v = u + at$

$$v = 5 + (-10) \times 1$$
$$v = -5$$

The horizontal velocity is still 4 ms^{-1}. It does not change.



Angle of flight:

$$\tan A = \frac{5}{4}$$
$$A = 51.3^\circ \text{ (3s.f.)}$$

So, after 1 second, the direction of flight is 51.3° (3 s.f.) below the horizontal.

MEI Mechanics 1

Projectiles

Section 2: General equations

Notes and Examples

These notes contain subsections on

- **The equation of the trajectory of a projectile**
- **Further examples**

The equation of the trajectory of a projectile

If you eliminate the t variable from the equations for vertical displacement (y) and horizontal displacement (x), you get a Cartesian equation, which gives you the projectile's path through the air.

The path equation is very useful. It gives you a way of identifying each point on the trajectory easily.

Example 1

A particle is projected from a point 1 m above the ground, with initial velocity of 10 ms^{-1} at an angle θ above the horizontal, where $\cos\theta = 0.8$ and $\sin\theta = 0.6$. The origin is taken to be the point on the ground directly below the point of projection.

(i) Find equations for the horizontal and vertical positions, x m and y m, in terms of t.

(ii) Find the Cartesian equation of the trajectory of the particle.

(iii) Find the value of y when x is 2.

(iv) Find the possible values of x when y is 2.

Solution

(i) The initial horizontal speed of the particle is $10\cos\theta = 10 \times 0.8 = 8$

Horizontally the speed is constant

so $x = 8t$

The initial vertical speed of the particle is $10\sin\theta = 10 \times 0.6 = 6$

The vertical displacement is given by
$$\begin{aligned} s &= ut + \tfrac{1}{2}at^2 \\ &= 6t + \tfrac{1}{2} \times -10t^2 \\ &= 6t - 5t^2 \end{aligned}$$

The initial height is 1 m

so $y = 1 + 6t - 5t^2$

(ii) $x = 8t \Rightarrow t = \dfrac{x}{8}$

Substituting into $y = 1 + 6t - 5t^2$:

$$y = 1 + \frac{6x}{8} - 5\left(\frac{x}{8}\right)^2$$

$$y = 1 + \frac{3}{4}x - \frac{5}{64}x^2$$

(iii) When $x = 2$, $y = 1 + \frac{3}{4} \times 2 - \frac{5}{64} \times 2^2$

$$= 1 + \frac{3}{2} - \frac{5}{16}$$

$$= 2.1875$$

(iii) When $y = 2$, $2 = 1 + \frac{3}{4}x - \frac{5}{64}x^2$

$$\frac{5}{64}x^2 - \frac{3}{4}x + 1 = 0$$

$$5x^2 - 48x + 64 = 0$$

$$(x-8)(5x-8) = 0$$

$$x = 8 \text{ or } x = 1.6$$

In the same way, it is possible to find the general equation for the path of a projectile in terms of its initial velocity V and angle of projection α:

$$y = x\tan\alpha - \frac{gx^2}{2V^2}\left(1 + \tan^2\alpha\right)$$

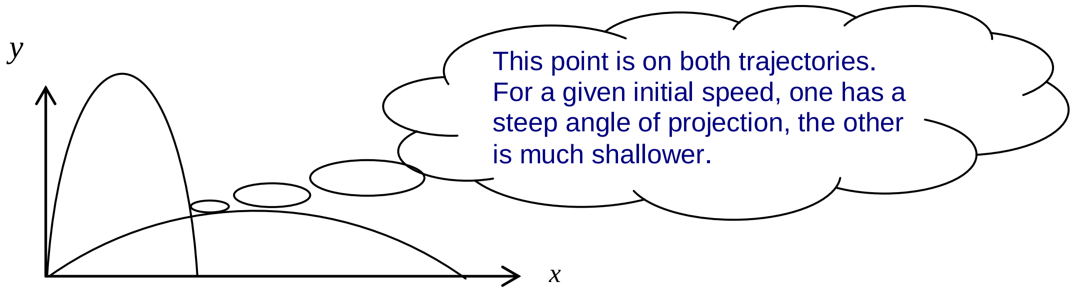
Notice that this equation is quadratic in both α and x. This means that for each possible point on a projectile's path, for every initial velocity there are two angles which will result in the projectile passing through the point, except at maximum range.
(for a quadratic equation, for all y values, except the maximum (or minimum) point, there are two possible x values – think about the graph of a quadratic)

The derivation of the path equation $y = x\tan\alpha - \frac{gx^2}{2V^2}\left(1 + \tan^2\alpha\right)$ is given on page 127. To follow it fully, you need to know from Core 4 work that $1 + \tan^2\alpha = \sec^2\alpha = \frac{1}{\cos^2\alpha}$ (however, don't worry if you haven't covered this yet – you do not need to know this derivation).

You should **NOT** attempt to learn or quote this equation. If you are asked to find the equation of a trajectory, you will be asked to eliminate t from the equations for x and y, as in Example 1 above.

There are usually two ways of "hitting" a point with a given initial speed.
One path has a low trajectory and a short time of flight, whereas the other has a much higher trajectory and a longer time of flight.

MEI M1 Projectiles Section 2 Notes and Examples



In all this work you have assumed ideal conditions, with, in particular, no air resistance. In reality this can be quite important as it can have various effects on the particle, as well as just slowing it down e.g. spinning and turning.

The ***Projectiles interactive spreadsheet*** allows you to view the path of a projectile for which you can set the initial velocity, the angle of projection and the initial height. Instructions are available.

The ***Projectiles application*** provides four different activities: two investigations and two worked examples. Instructions are available.

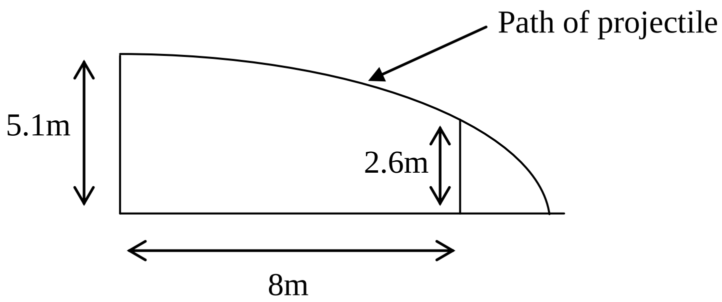
Further Examples

Example 2
A boy throws a ball horizontally from a point 5.1 m above the horizontal ground.

(i) What is the minimum speed at which the ball must be thrown to clear a fence 2.6 m high at a horizontal distance of 8 m from the point of projection?

(ii) Find the distance beyond the fence at which the ball strikes the ground if it is projected at this speed?

Solution



(i) In the time the ball travels 8 m horizontally, it must fall a maximum distance of 5.1 – 2.6 = 2.5 m if it is to clear the fence.

Considering the vertical motion:
The time to fall 2.5 m:

$u = 0$

$a = 9.8\text{ ms}^{-2}$

$s = 2.5\text{ m}$

$t = ?$

$$s = ut + \frac{1}{2}at^2$$

$$2.5 = 4.9t^2$$

$$t = 0.714\text{ seconds (3 s.f.)}$$

So the ball travels 8 m horizontally in 0.714 s,

so its initial horizontal speed is $\frac{8}{0.714} = 11.2\ \text{ms}^{-1}$ (3 s.f.), or greater.

(ii) The time for the ball to fall to ground level is found as above, except that now s = 5.1 m, instead of 2.5 m.

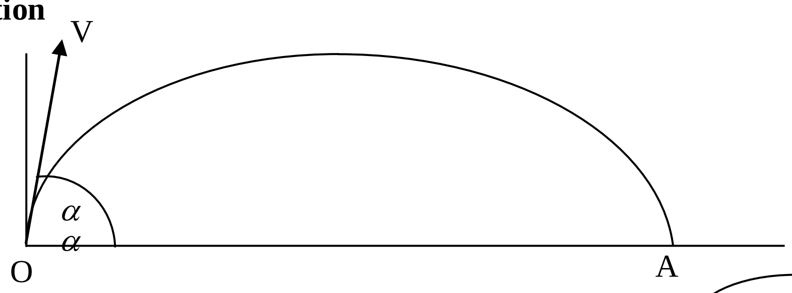
So $5.1 = 4.9t^2 \Rightarrow t = \sqrt{\frac{5.1}{4.9}} = 1.02\text{s}$ (3 s.f.)

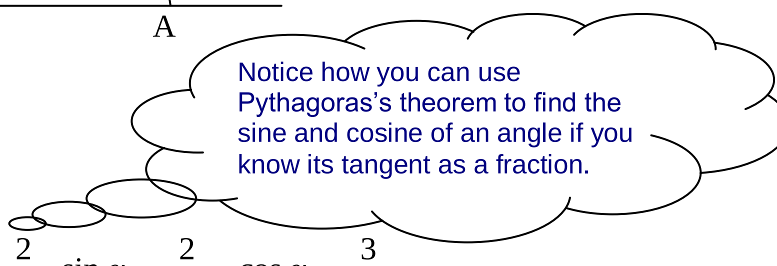
In 1.02 seconds the ball will travel $1.02 \times 11.2 = 11.42\text{m}$ (4 s.f.) horizontally, so it will land 11.42 - 8 = 3.42 m (3 s.f.) beyond the fence.

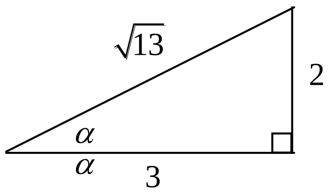
Example 3

A projectile is fired with an initial speed V from a point O on a horizontal plane, the angle of elevation being $\tan^{-1}\left(\frac{2}{3}\right)$. The particle returns to the plane at a point A. Find the distance OA, in terms of V. Show that the same point could have been reached by firing the particle at the same speed from O, but at an angle of elevation $\tan^{-1}\left(\frac{3}{2}\right)$. Find, in terms of V, the difference in the times of flights of the two trajectories.

Solution







$\tan\alpha = \frac{2}{3}$, $\sin\alpha = \frac{2}{\sqrt{13}}$, $\cos\alpha = \frac{3}{\sqrt{13}}$

To find the time of flight, since the particle lands on the same level as it started, we can find the time to maximum height and then double it:

Vertically:

$u = V\sin\alpha = \frac{2}{\sqrt{13}}V$

$v = 0$ (at max height)

$a = -9.8$

$t = ?$

$v = u + at$

$0 = \frac{2V}{\sqrt{13}} - 9.8t$

$t = \frac{2V}{9.8\sqrt{13}}$

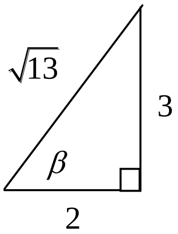
The time of flight is double this,

so the time of flight is $t = \frac{4V}{9.8\sqrt{13}} = 0.113V$ seconds (3 s.f)

The horizontal range is the value of x when t takes the above value.

$$\begin{aligned} x &= Vt\cos\alpha \\ &= V \times \frac{4V}{9.8\sqrt{13}} \times \frac{3}{\sqrt{13}} \\ &= \frac{12V^2}{9.8\times 13} \\ &= 0.094V^2 \text{ (3 d.p.)} \end{aligned}$$

If the particle is projected at the same speed, V, but at angle β, such that $\tan\beta = \frac{3}{2}$, then:



$$\tan\beta = \frac{3}{2},\ \sin\beta = \frac{3}{\sqrt{13}},\ \cos\beta = \frac{2}{\sqrt{13}}$$

To find the time of flight, find the time to maximum height and then double it, as before:

Vertically:

$u = V\sin\alpha = \frac{3}{\sqrt{13}}V$

$v = 0$ (at max height)

$a = -9.8$

$t = ?$

$$\begin{aligned} v &= u + at \\ 0 &= \frac{3V}{\sqrt{13}} - 9.8t \\ t &= \frac{3V}{9.8\sqrt{13}} \end{aligned}$$

The time of flight is double this,

So the time of flight is $\frac{6V}{9.8\sqrt{13}} = 0.170V$ seconds (3 s.f).

The horizontal range is the value of x when t takes the above value.

$$\begin{aligned} x &= Vt\cos\alpha \\ &= V \times \frac{6V}{9.8\sqrt{13}} \times \frac{2}{\sqrt{13}} \\ &= \frac{12V^2}{9.8\times 13} \\ &= 0.094V^2 \text{ (3 d.p.)} \end{aligned}$$

So the ranges are the same.

The difference in the times of flight is $\frac{6V}{9.8\sqrt{13}} - \frac{4V}{9.8\sqrt{13}} = \frac{2V}{9.8\sqrt{13}} = 0.0566V$ seconds (3 s.f.)

MEI Mechanics 1

Forces and motion in two dimensions

Section 1: Resultant forces

Notes and Examples

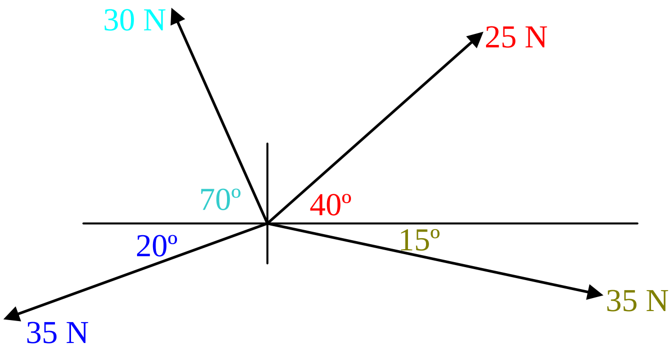
These notes contain subsections on

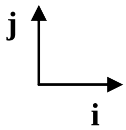
- **Resolving forces**
- **Problems involving slopes**
- **Particles**
- **Three forces in equilibrium**

Resolving forces

Example 1

Resolve each force below into a horizontal and vertical component and decide whether the system is in equilibrium.





Solution

Force	Horizontal component (**i**)	Vertical component (**j**)
25 N	25cos 40 = 19.15	25sin 40 = 16.07
35 N	35cos 15 = 33.81	-35sin 15 =-9.06
35 N	-35cos20 = -32.89	-35sin 20 = -11.97
30 N	-30cos 70 =-10.26	30 sin 70 = 28.19
Resultant (total of each component)	9.81	23.23

Be careful to get the signs right. A table, like this one, can help you to avoid mistakes.

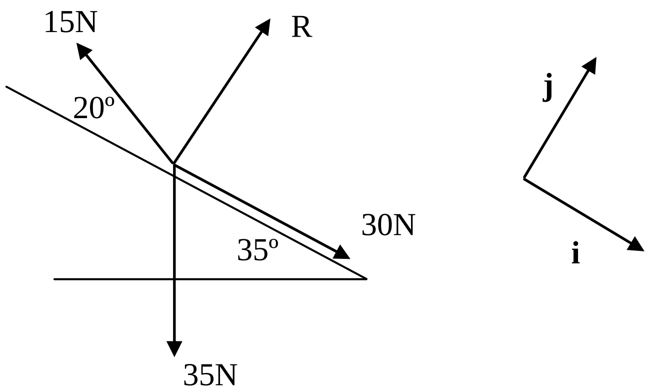
So there is a resultant force of (9.81**i** + 23.2**j**) N (3 s.f.) and the system is therefore not in equilibrium (if it were in equilibrium, both components would have to be 0).

MEI M1 Forces in 2D Section 1 Notes and Examples

Problems involving slopes

In situations involving particles on slopes, like in example 2 below, the problem is always simplified by choosing **i** and **j** to be parallel and perpendicular to the slope, so that the normal reaction is in the **j** direction and any forces parallel to the slope are in the **i** direction.

Example 2
An object sliding on a slope is subjected to the forces shown in the diagram below. Resolve all the forces parallel and perpendicular to the slope to find the resultant force and the magnitude of the normal reaction, R N.



Solution

Force	Parallel (**i**)	Perpendicular (**j**)
15N	-15cos 20º = -14.10	15 sin 20º = 5.13
RN	0	R
30N	30	0
35N	35sin 35° = 20.08	-35cos 35° = -28.67
Resultant	35.98	R – 23.54

Since the particle is sliding on the slope, the resultant force in the **j** direction must be 0, so R = 23.5**j** (3s.f.), so the magnitude of the normal reaction is 23.5 N.

The resultant force is 36.0**i** N (3s.f.)

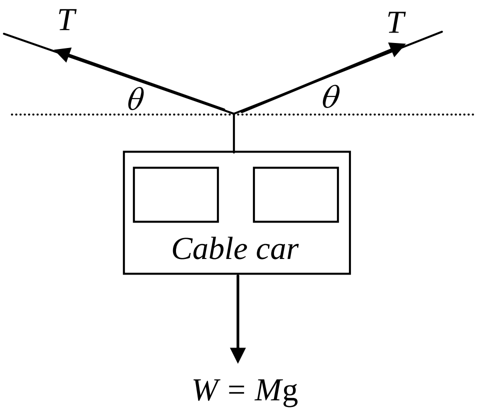
For some practice in resolving on a slope, try the interactive spreadsheet ***Resolving***. There is an example showing how the weight is resolved into components parallel to and perpendicular to the slope, and a problem for you to try, in which you can vary the angle of the slope and the weight of the object.

MEI M1 Forces in 2D Section 1 Notes and Examples

Particles

Note that all of the forces which are acting on a particle have lines of action which go through ONE point. When we talk about objects as particles, we are making a modelling assumption that the object can be considered as having all of its mass concentrated at one point, with any forces acting upon it acting through that point. If this does not happen, the object can rotate. You will learn how to model turning forces in Mechanics 2, should you choose to do it.

***Question mark*, page 130**
It is worth noting that any particle suspended on a wire, **must** cause the wire to sag. The vertical component of the tensions in the wire must balance the weight of the particle. It is impossible to suspend a particle from a horizontal wire, no matter how tight it is, as it could have no vertical component of tension to balance the weight of the suspended particle. This links to the cable car question on page 130. The vertical component of the tensions in the cable must balance the weight of the cable car.



Resolving vertically,

$$2T\sin\theta = W \Rightarrow T = \frac{W}{2\sin\theta}$$

This means if $\sin\theta < \frac{1}{2}$, $T > W$

$\sin 30^\circ = \frac{1}{2}$, so $T > W$ for $\theta < 30^\circ$

For a further example of an object in equilibrium, look at the Flash resource ***Forces acting on a box***.

Three forces in equilibrium

In situations where there are three forces acting in equilibrium, trigonometry can be used because the vectors representing the three forces must form a closed triangle. Method 2 for example 7.2 on page 137 makes use of the sine rule. Example 3 below also uses the sine rule and Example 4 uses the cosine rule.

Example 3
A crate of mass 300 kg is hanging from a rope. Two dockers are pulling it downwards and to one side using a second rope, with a combined force of 1000 N. If the rope the crate is hanging from makes an angle of 15° to the vertical and the system is in equilibrium, find

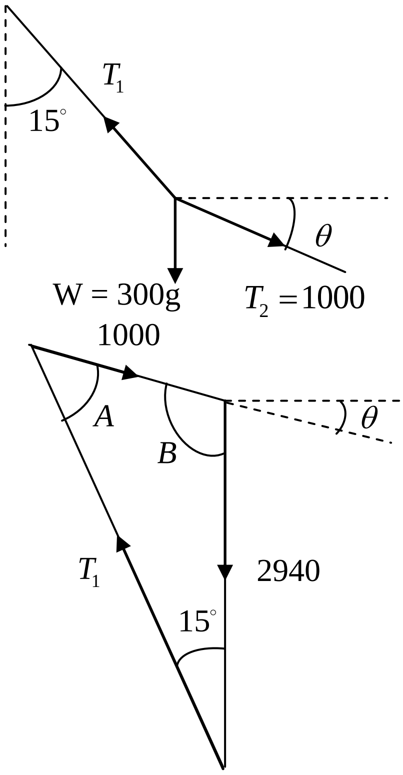
(i) the angle which the second rope makes to the horizontal,
(ii) the magnitude of the tension in the rope from which the crate is hanging.

Solution

(i) There are 3 forces acting on the crate:

T_1, at 15° to the vertical
$300g$ = 2940 N vertically downwards
$T_2 = 1000$ N at θ to the horizontal

These forces are in equilibrium, so they can be represented as a closed triangle of forces.
Using the sine rule:



$$\frac{\sin 15^\circ}{1000} = \frac{\sin A}{2940} \Rightarrow \sin A = \frac{2940 \times \sin 15^\circ}{1000}$$

$$\Rightarrow A = 49.55^\circ \text{ or } (180 - 49.55) = 130.45^\circ$$

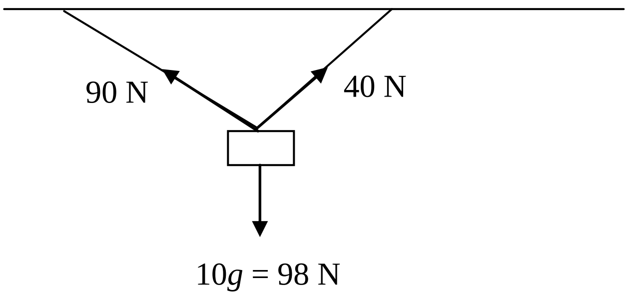
Remember that the sine rule sometimes gives two possible answers. You may need to decide which is correct depending on the context of the question

Since the dockers are pulling with a downward component, A must be acute (less than 90°) so $A = 49.55^\circ$.
So $B = 180 - A - 15 = 180 - 49.55 - 15 = 115.45^\circ$
So $\theta = 90 - (180 - B) = 90 - 64.55 = 25.45^\circ$

(ii) Using the sine rule, $\dfrac{1000}{\sin 15} = \dfrac{T_1}{\sin 115.45} \Rightarrow T_1 = 3490$ N (3s.f.)

Example 4

A mass of 10 kg is suspended in equilibrium from two cables, as shown below. One cable has a tension of 90 N, the other of 40 N. What angle does each of these cables make to the vertical?



Solution

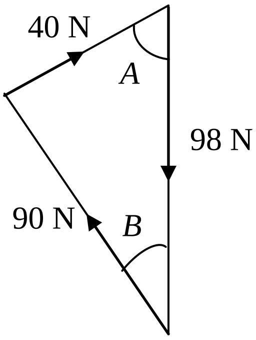
Using the cosine rule:

$$90^2 = 40^2 + 98^2 - 2 \times 40 \times 98 \cos A$$

$$\Rightarrow \frac{40^2 + 98^2 - 90^2}{2 \times 40 \times 98} = \cos A \Rightarrow A = 66.7^\circ \text{(3.s.f.)}$$

$$40^2 = 90^2 + 98^2 - 2 \times 90 \times 98 \cos B$$

$$\Rightarrow \frac{90^2 + 98^2 - 40^2}{2 \times 90 \times 98} = \cos B \Rightarrow B = 24.1^\circ \text{(3.s.f.)}$$



so the 40 N force acts at 66.7° to the vertical and the 90 N force at 24.1° to the vertical (3 s.f.).

You can look investigate another equilibrium problem using the Flash resource ***Pulleys***.

MEI Mechanics 1

Forces and motion in two dimensions

Section 2: Newton's second law

Notes and examples

These notes contain subsections on

- The vector form of Newton's 2nd law
- Further examples

The vector form of Newton's 2nd law

If information is given in component form, it is possible simply to apply Newton's 2nd Law in vector form to the problem.

Remember that when working in vector form, force and acceleration are both vectors, but mass is not. So Newton's 2nd law can be written as

$$\mathbf{F} = m\mathbf{a}$$

or in handwriting: $\underline{F} = m\underline{a}$

This is shown in the following example.

Example 1
Two forces of $3\mathbf{i} + 2\mathbf{j}$ and $5\mathbf{i} - 3\mathbf{j}$ act on a particle of mass 10 kg.
(i) What is the acceleration of the particle?
(ii) What additional force must act on the particle to give it an acceleration of $2\mathbf{i} + \mathbf{j}$?

Solution
(i) The resultant force on the particle is $(3\mathbf{i} + 2\mathbf{j}) + (5\mathbf{i} - 3\mathbf{j}) = 8\mathbf{i} - \mathbf{j}$.
Newton's 2nd law: $8\mathbf{i} - \mathbf{j} = 10\mathbf{a}$
$\mathbf{a} = 0.8\mathbf{i} - 0.1\mathbf{j}$

Notice that the acceleration is a vector. Remember that in handwriting you must underline vectors.

(ii) Let the additional force be $\mathbf{F}$.
$8\mathbf{i} - \mathbf{j} + \mathbf{F} = 10(2\mathbf{i} + \mathbf{j})$
$\mathbf{F} = 20\mathbf{i} + 10\mathbf{j} - (8\mathbf{i} - \mathbf{j})$
$\mathbf{F} = 12\mathbf{i} + 11\mathbf{j}$

In the next chapter you will learn to work with accelerations in vector form. You can apply the constant acceleration formulae to situations in two or three dimensions by using accelerations, velocities and displacements in vector form.

MEI M1 Forces in 2D Section 2 Notes and Examples

Further examples

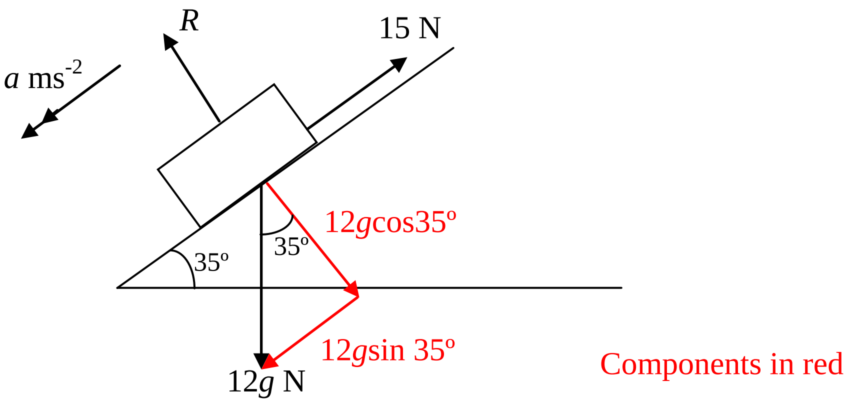
In a situation where there are forces acting on an object in two dimensions, and there is motion, then you should consider the components of the forces in the direction of motion, and in the direction perpendicular to the motion.

This makes the calculations simpler: in the direction of motion you can apply Newton's 2nd law $F = ma$, and in the direction perpendicular to the motion there is no motion and therefore no resultant force (this is a special case of Newton's 2nd law).

Always start by drawing a good-sized diagram showing all the forces and angles.

Example 2
A block of mass 12 kg is on a slope of angle 35°. It experiences a resistance force of 15 N as it slides down the slope. Find
(i) the reaction force
(ii) the acceleration of the block.

Solution



(i) Using Newton's second law perpendicular to the slope:

$$R - 12g\cos 35° = 0$$
$$R = 12 \times 9.8\cos 35° = 96.3 \text{ (3 s.f.)}$$

There is no acceleration in the direction perpendicular to the slope.

The reaction force is 96.3 N (3 s.f.).

(ii) Using Newton's second law down the slope:

$$12g\sin 35° - 15 = 12a$$
$$a = \frac{12 \times 9.8\sin 35° - 15}{12} = 4.37 \text{ (3 s.f.)}$$

The acceleration in the direction parallel to the slope is a ms^{-2}.

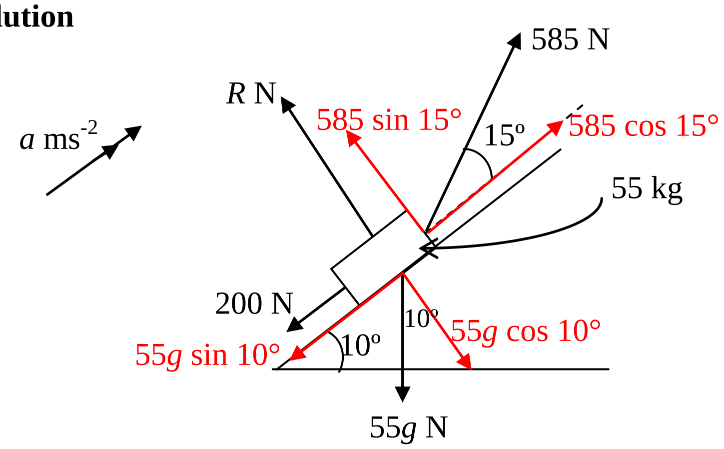
So the acceleration is 4.37 ms^{-2} down the slope.

Example 3
An arctic explorer is pulling a sledge containing her provisions, a total mass of 55 kg, up a slight incline of 10º. The rope she is pulling with makes an angle of 15º with the slope, and she pulls with a force of 585 N. The ice causes a resistance to motion of 200 N.

(i) Calculate the acceleration of the sledge.

(ii) If she starts from rest, how long does it take her to reach a speed of 2 ms^{-1}?

Solution



Components in red

(i) Using Newton's second law parallel to the slope:

$$585\cos 15° - 55g\sin 10° - 200 = 55a$$

$$a = \frac{585\cos 15° - 55\times 9.8\sin 10° - 200}{55} = 4.936 \text{ (4 s.f.)}$$

The acceleration is 4.94 ms^{-2} (3 s.f.)

(ii)

$u = 0$	$v = u + at$
$a = 4.936$	$2 = 4.936t$
$v = 2$	$t = 0.405$ (3 s.f.)
$t = ?$	

She takes 0.405 s (3 s.f.) to reach a speed of 2 ms^{-1}.

MEI Mechanics 1

General motion

Section 1: Using calculus

Notes and Examples

These notes contain subsections on:

- Using differentiation
- Using integration

Using differentiation

If you are given the formula for the position of a particle in terms of t, then:

- to find its velocity at any instant, you differentiate the position with respect to time (t) and substitute in the appropriate value for t.

- to find its acceleration at any instant, you differentiate the velocity with respect to time (t) and substitute in the appropriate value for t.

Example 1

The position, s m, of a particle after t seconds is given by $s = t^3 - 5t^2 + 7t - 3$.

(a) Find (i) the velocity
(ii) the acceleration
of the particle after 3 seconds.

(b) Find t when (i) $v = 5$ ms^{-1}
(ii) $a = 6$ ms^{-2}.

Solution

(a) (i) The velocity is given by $v = \dfrac{ds}{dt} = 3t^2 - 10t + 7$

When $t = 3$, $v = 3 \times 3^2 - 10 \times 3 + 7$
$= 4$

The velocity of the particle is 4 ms^{-1}.

(ii) The acceleration is given by $a = \dfrac{dv}{dt} = \dfrac{d^2s}{dt^2} = 6t - 10$

When $t = 3$, $a = 6 \times 3 - 10$
$= 8$

The acceleration of the particle is 8 ms^{-2}.

(b) (i) From (a), $v = 3t^2 - 10t + 7$.

When $v = 5$: $\quad 3t^2 - 10t + 7 = 5$

$$3t^2 - 10t + 2 = 0$$

$$t = 3.12 \text{ or } t = 0.214 \text{ (3 s.f.)}$$

(ii) From (a), $a = 6t - 10$

When $a = 6$: $\quad 6t - 10 = 6$

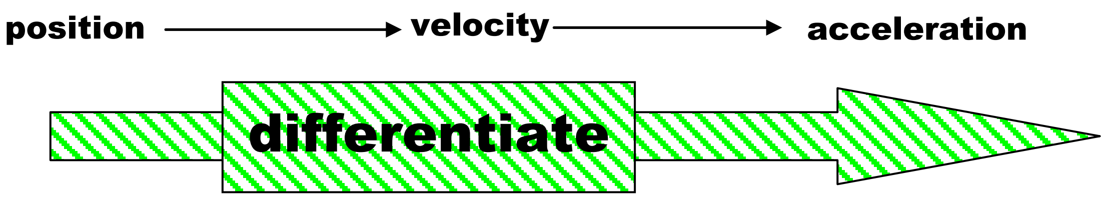
$$t = \frac{16}{6} = 2.67 \text{ (3 s.f.)}$$

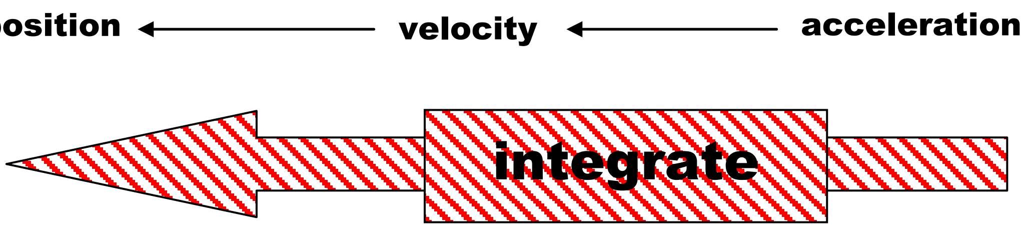
Using integration

If you are given the formula for the acceleration of a particle in terms of t, then:

- To find its velocity at any instant, you integrate the acceleration with respect to time (t) and substitute in the appropriate value for t.
- To find its position at any instant, you integrate the velocity with respect to time (t) and substitute in the appropriate value for t.

This can be summarised by the diagram below:-





Example 2

A particle, initially at rest at the point where $s = 3$, has an acceleration at time t seconds given by $a = t^3 - 2t^2$.

Find expressions for its velocity and position at time t.

Solution

$$a = \frac{\mathrm{d}v}{\mathrm{d}t} \Rightarrow v = \int t^3 - 2t^2 \mathrm{d}t \Rightarrow v = \frac{t^4}{4} - \frac{2t^3}{3} + c$$

To find the value of c, use the information in the question which states that the particle is initially at rest, so when $t = 0$, $v = 0$.
Substituting these into the equation for v gives $c = 0$

so $v = \frac{t^4}{4} - \frac{2t^3}{3}$

To find an expression for s, integrate again, and use the information from the question that $s = 3$ when $t = 0$ to find the constant of integration.

$$s = \int v \mathrm{d}t = \int \frac{t^4}{4} - \frac{2t^3}{3} \mathrm{d}t = \frac{t^5}{20} - \frac{t^4}{6} + k\ .$$

Since $s = 3$ when $t = 0$, $k = 3$

so $s = \frac{t^5}{20} - \frac{t^4}{6} + 3$

You can see further examples of the use of differentiation and integration when working with expressions for displacement, velocity and acceleration, using the Flash resource ***General motion***.

MEI Mechanics 1

General motion

Section 2: Motion in two and three dimensions

Notes and examples

The key to this work is to remember when to differentiate and when to integrate.
If you are given the position vector, you differentiate once to find the velocity vector (each component, of course) and differentiate again to find the acceleration.

It tends to be slightly harder to find the velocity vector or position vector given the acceleration vector. You integrate in these cases, but remember the constants of integration. These must be worked out at each stage, before you go onto the next.

Example 1
The force acting on a particle of mass 5 kg is given as $(20\mathbf{i} + 15t\mathbf{j})$ N. If its initial velocity is $(3\mathbf{i} + 25\mathbf{j})$ ms^{-1}, find its position after 3 seconds, given that its position after 1 second was $(8\mathbf{i} + 7\mathbf{j})$ m.

Solution
$\mathbf{F} = m\mathbf{a}$ (Newton's second law), so its acceleration is $\frac{1}{5}(20\mathbf{i} + 15t\mathbf{j}) = 4\mathbf{i} + 3t\mathbf{j}$

Integrating each component gives its velocity as

$$(4t + c)\mathbf{i} + \left(\frac{3}{2}t^2 + d\right)\mathbf{j}$$

You need to find the values of the integration constants, c and d.
Its velocity at $t = 0$ was $3\mathbf{i} + 25\mathbf{j}$. This means $c = 3$ and $d = 25$.

Therefore the velocity is $(4t + 3)\mathbf{i} + \left(\frac{3}{2}t^2 + 25\right)\mathbf{j}$

Integrating each component gives its position vector as

$$\mathbf{r} = (2t^2 + 3t + e)\mathbf{i} + \left(\frac{1}{2}t^3 + 25t + f\right)\mathbf{j}$$

You need to find the values of the integration constants e and f.
The displacement after 1 second was $8\mathbf{i} + 7\mathbf{j}$. This means $e = 3$ and $f = -18.5$

Therefore $\mathbf{r} = (2t^2 + 3t + 3)\mathbf{i} + \left(\frac{1}{2}t^3 + 25t - 18.5\right)\mathbf{j}$

So, when $t = 3$, the position is $(30\mathbf{i} + 70\mathbf{j})$ m.