MEI Structured Mathematics

Practice Comprehension Task - 3

How far to shop?

If you draw a straight line on a map between the centre point of Nottingham and the centre point of Derby, you would draw the line very close to my house in Long Eaton.

Measured along this line, the centre of Nottingham from my house is 14 km and the centre of Derby from my house is 10 km. Yet I almost always go to shop in Nottingham.

Human beings are not always the most predictable of creatures and this is particularly so when people make decisions about where to live, where to work and where to shop. Some people live and work in the same village. Others travel for 50 km or more to work. Most people would like to be able to shop locally, but this is becoming increasingly difficult as small shops find it impossible to compete with their larger rivals. Thus it is common for people who live in villages to have to travel to neighbouring towns for even the most basic of shopping needs.

But where will you go for your shopping needs? Will you travel a short distance to a nearby town which has a medium-sized supermarket? Or will you travel further in order to get to a huge hypermarket with a seemingly infinite choice? Do you have a favourite hair salon where you always go even though it is 20 km away? On the other hand when it comes to repairs in the house, do you just find the nearest DIY store and get what you need as quickly as possible?

We are all different and there are as many decisions about shopping as there are shoppers. Yet when thousands of people are studied together, patterns of choice do begin to emerge. Studies of this kind are carried out by geographers, sociologists, statisticians and economists. Geographers, for example, observe the movement of people between towns and try to discern some kind of regularity or pattern. They may also try to explain how these patterns arise. Both in describing and explaining the patterns, geographers frequently use mathematical models.

Let us consider the simplest of models, such as that posed by the location of my house between Nottingham and Derby. Consider a person who lives at a point X on a straight line between 2 towns, A and B, as shown in figure 1.

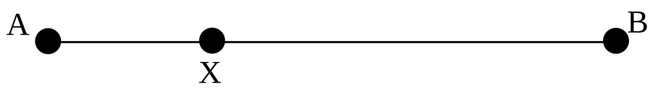

Fig. 1

They all have a choice as to which town they will visit to shop, and in this model they only have the two choices, A and B. If A and B are the same size then it would be reasonable to assume that if you live nearer to A than B, then you will go to A to shop. But if, say, B is considerably larger than A, then it might be preferable to go there to shop even though you live nearer to A.

In this model we will assume that the likelihood of a person travelling to a particular town is directly proportional to the population of the town (P) and also inversely proportional to the distance (d) to the town. This simple assumption can then be used to find the "breaking point", X, between the two towns.

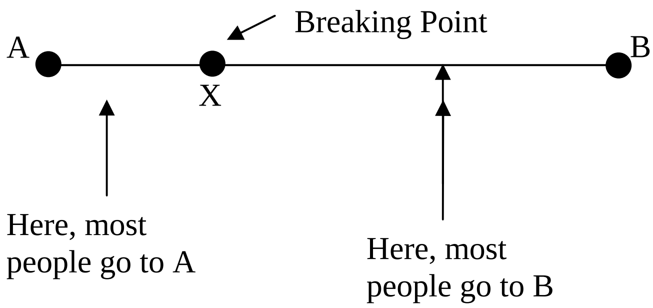

Fig. 2

If we let the population of town A be P_A and the population of town B be P_B and we let the distances between A and X and between B and X be d_{AX} and d_{BX} respectively, then, given that at the breaking point they are equally likely to go either way.

$$\frac{P_A}{d_{AX}} = \frac{P_B}{d_{BX}}$$

$$\Rightarrow d_{BX} = \frac{P_B}{P_A} \times d_{AX}$$

$$\Rightarrow d_{AB} - d_{AX} = \frac{P_B}{P_A} \times d_{AX}$$

where d_{AB} is the distance between towns A and B

$$\Rightarrow d_{AB} = \frac{P_B}{P_A} \times d_{AX} + d_{AX} = d_{AX}\left(1 + \frac{P_B}{P_A}\right)$$

$$\Rightarrow d_{AX} = \frac{d_{AB}}{\left(1 + \frac{P_B}{P_A}\right)}$$

This formula gives the distance of the breaking point from A.

As an example, consider Rugby and Coventry. The populations of these two towns are 61 100 and 299 300 respectively and the distance between them is 17.5 km. According to this model the distance of the breaking point from Rugby, d_{rx} is given by

$$d_{RX} = \frac{d_{RC}}{\left(1 + \frac{P_C}{P_R}\right)} = \frac{17.5}{\left(1 + \frac{299\,300}{61\,100}\right)} = 3.0 \text{ km (to 1 decimal place.)}$$

Hence the breaking point is 3.0 km from Rugby and 14.5 km from Coventry.

According to this model I am within the breaking distance for Nottingham, which is 14.4 km.

This model has some weaknesses. In particular, it probably underestimates the deterrent effect of distance. People prefer not to travel long distances to shop. W.J. Reilly attempted to create a better model by borrowing from Newton's Law of Universal Gravitation.

In this revised model, known as Reilly's Law of Retail Gravitation, we will assume that the likelihood of a person travelling to a particular town is directly proportional to the population of the town (P) and also inversely proportional to the square of the distance (d) to the town.

Thus we obtain the following

$$\frac{P_A}{(d_{AX})^2} = \frac{P_B}{(d_{BX})^2}$$

and this leads to a revised formula for the breaking point as

$$d_{AX} = \frac{d_{AB}}{\left(1 + \sqrt{\frac{P_B}{P_A}}\right)}$$

For Rugby and Coventry this would give

$$d_{RX} = \frac{d_{RC}}{\left(1 + \sqrt{\frac{P_C}{P_R}}\right)} = \frac{17.5}{\left(1 + \sqrt{\frac{299300}{61100}}\right)} = 5.4 \text{ km (to 1 decimal place.)}$$

Thus, by this revised model, people living further away from Rugby than the original breaking point would travel to Rugby rather than Coventry.

Even this is a very simple model, assuming that there are only two towns from which to choose. It also assumes that the way to define the size of the town is by its population - since this model refers to shopping it might be a better definition of size to use the total number of shops or the floor area of all shops.

Also, because of the use of car ownership it may be that distance has become less of a factor in deciding where to shop (and this includes, of course, the ease of parking, or the efficiency of the public transport system). This could be recognised by reducing the power of d_{ab} in the Law of Retail Gravitation to something like 1.5.

A further revised model may therefore be $d_{AX} = \frac{d_{AB}}{\left(1 + \left(\frac{P_B}{P_A}\right)^{\frac{2}{3}}\right)}$

It is amazing to find that amongst the seemingly random habits of human beings, it is often possible to discern patterns. Even when we consider preferences for shopping we find that it is mathematical modelling that provides us with some interesting and powerful tools for analysis. You might like to carry out an investigation along these lines in your own locality!

Questions

1 In line 42 is given the formula for the distance of the breaking point, d_{ax}, from town A. Write the equivalent formula for d_{bx} and add them together to confirm your formula. [4]

2 In line 50 the breaking distance for my house from Nottingham is given as 14.4 km. Show that this is so, given that the populations of Nottingham and Derby are 300 000 and 200 000. respectively, to 1 significant figure. [2]

3 Derive the formula for the distance of the breaking point using Reilly's Law of Retail Gravitation in line 60. [4]

4 Find the distance of the breaking point between Nottingham and Derby using Reilly's Law of Retail Gravitation and comment on your result, referring to where my house is situated. [3]

5 You are given the following data with respect to two towns, Rugby and Leicester.

Population of Rugby	61 100
Breaking point	9.3 km from Rugby.
Distance between Rugby and Leicester	31.5 km

Calculate the population of Leicester using Reilly's Law of Retail Gravitation. [3]

6 Suggest an adaptation of Reilly's Law of Retail Gravitation which might be necessary for a country where travel is largely on foot. [2]

Answers.

1		$d_{ax}=\dfrac{d_{ab}}{\left(1+\dfrac{P_b}{P_a}\right)}\Rightarrow d_{bx}=\dfrac{d_{ab}}{\left(1+\dfrac{P_a}{P_b}\right)}$ $d_{ax}+d_{bx}=\dfrac{d_{ab}}{\left(1+\dfrac{P_b}{P_a}\right)}+\dfrac{d_{ab}}{\left(1+\dfrac{P_a}{P_b}\right)}$ $\Rightarrow d_{ax}+d_{bx}=\dfrac{P_a d_{ab}}{\left(P_a+P_b\right)}+\dfrac{P_b d_{ab}}{\left(P_a+P_b\right)}=\dfrac{\left(P_a+P_b\right)}{\left(P_a+P_b\right)}d_{ab}$ $\Rightarrow d_{ax}+d_{bx}=d_{ab}$	B1 M1 M1 A1 **4**	 For sum Adding
2		$d_{nx}=\dfrac{d_{dn}}{\left(1+\dfrac{P_d}{P_n}\right)}\Rightarrow d_{nx}=\dfrac{24}{\left(1+\dfrac{200\,000}{300\,000}\right)}=24\times\dfrac{3}{5}=14.4$	M1 A1 **2**	
3		$\dfrac{P_a}{\left(d_{ax}\right)^2}=\dfrac{P_b}{\left(d_{bx}\right)^2}$ $\Rightarrow\left(d_{bx}\right)^2=\dfrac{P_b}{P_a}\times\left(d_{ax}\right)^2\Rightarrow d_{bx}=d_{ax}\sqrt{\dfrac{P_b}{P_a}}$ $\Rightarrow d_{ab}-d_{ax}=d_{ax}\sqrt{\dfrac{P_b}{P_a}}$ $\Rightarrow d_{ab}=d_{ax}\sqrt{\dfrac{P_b}{P_a}}+d_{ax}=d_{ax}\left(1+\sqrt{\dfrac{P_b}{P_a}}\right)$ $\Rightarrow d_{ax}=\dfrac{d_{ab}}{\left(1+\sqrt{\dfrac{P_b}{P_a}}\right)}$	M1 A1 A1 A1 **4**	
4		$d_{nx}=\dfrac{d_{dn}}{\left(1+\sqrt{\dfrac{P_d}{P_n}}\right)}\Rightarrow d_{nx}=\dfrac{24}{1.816}=13.2$ My house was within the breaking point but is not now!	M1 A1 B1 **3**	
5		$d_{ax}=\dfrac{d_{ab}}{\left(1+\sqrt{\dfrac{P_b}{P_a}}\right)}\Rightarrow 9.3=\dfrac{31.5}{\left(1+\sqrt{\dfrac{P_l}{61100}}\right)}$ $\Rightarrow 1+\sqrt{\dfrac{P_l}{61100}}=\dfrac{31.5}{9.3}=3.387\Rightarrow\sqrt{\dfrac{P_l}{61100}}=2.387$ $\Rightarrow P_l=61100\times 2.387^2\Rightarrow P_l\approx 348100$	M1 A1 A1 **3**	
6		Change power of *d* Increase power of *d* to e.g. 3	B1 B1 **2**	