C4: Comprehension question

Rationale

The aim of the comprehension question is to foster an appreciation among students that, in learning mathematics, they are acquiring skills which transcend the particular items of the specification content which have made up their course.

The objectives are that candidates should be able to:

(i) read and comprehend a mathematical argument or an example of the application of mathematics;

(ii) respond to a synoptic piece of work covering ideas permeating their whole course;

(iii) appreciate the relevance of particular techniques to real-world problems.

Description and conduct

The examination for Section C of *Applications of Advanced Mathematics (C4)* includes a comprehension question on which candidates are expected to take no more than 40 minutes. The question takes the form of a written article followed by questions designed to test how well candidates have understood it.

Candidates are allowed to bring standard English dictionaries into the examination and those for whom English is a second language are also allowed a translation dictionary. However care will be taken in preparing the question to ensure that the language is readily accessible.

Content

By its nature, the content of the written piece of mathematics cannot be specified in the detail of the rest of the specification. However knowledge of GCSE and *C1*, *C2* and *C3* will be assumed, as well as the content of the rest of this unit. Candidates are expected to be aware of ideas concerning accuracy and errors. The written piece may follow a modelling cycle and in that case candidates will be expected to recognise it. No knowledge of mechanics will be assumed.

Planetary Systems

Introduction

During most of the 18th century, just five planets, apart from Earth, were known: Mercury, Venus, Mars, Jupiter and Saturn.

In 1766, Titius of Wittenburg discovered that the distances from the Sun of the known planets formed an interesting mathematical sequence. This was published six years later by Johann Bode and became known as Bode's Law. Shortly afterwards, in 1781, the planet Uranus was first observed and it too obeyed Bode's Law.

Before going on to look at Bode's Law, two points are worth noting.

(i) In this article, distances are given in metric units and so the numbers differ by a constant scale factor from those which were used in the 18th century.

(ii) The paths of planets are ellipses rather than circles. Fig. 1 shows such a path (or orbit) around a star. Notice that the star is not at the centre of the ellipse.

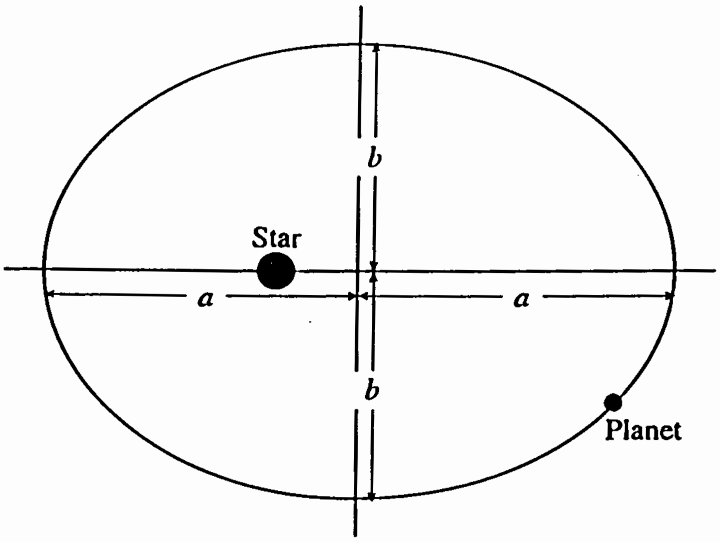

Fig. 1

A measure of how much an ellipse differs from a circle is called its *eccentricity*. The eccentricity of an ellipse is given by e ($0 \leqslant e < 1$) in the equation

$$e^2 = 1 - \frac{b^2}{a^2}$$

where a and b are the lengths shown in Fig. 1. A circle has eccentricity 0.

With the exception of Pluto, the planets in our solar system have nearly circular orbits. Consequently, in this article the eccentricity is ignored. The orbits are taken to be circular with the Sun at the centre.

Bode's Law

Take the geometric progression 1, 2, 4, ... , 64 and introduce 0 as a first term:

$$0, 1, 2, 4, \ldots, 64.$$

Now multiply each term by $4\frac{1}{2}$ and then add 6 to obtain Bode's numbers:

$$6, 10.5, 15, 24, \ldots, 294.$$

These numbers, multiplied by 10^7, give good approximations for the distances in kilometres of the planets Mercury to Uranus from the Sun, as shown in Table 2.

	Planet	Bode's number	Distance from the Sun ($\times 10^7$ km)
1	Mercury	6	5.8
2	Venus	10.5	10.8
3	Earth	15	15.0
4	Mars	24	22.8
5		42	
6	Jupiter	78	77.8
7	Saturn	150	142.7
8	Uranus	294	286.9

Table 2

It was immediately noticed that there seemed to be a planet missing, and a group of astronomers were assigned the task of finding it; they were nicknamed "The Celestial Police". In the early years of the 1800s four bodies, rather than one, were found at about the distance from the Sun predicted by Bode's Law. All of these were small and they became known as *asteroids*. Since then several thousand asteroids have been discovered, many of them little more than large rocks, occupying a belt between Mars and Jupiter.

In 1841, the planet Neptune was discovered at a distance of 449.8×10^7 km from the Sun, a distance which did not agree with that predicted by extending Bode's Law.

In 1930, Pluto was discovered and it also failed to conform to Bode's Law. However, Pluto differs from the other planets in so many respects that most astronomers no longer call it a planet. As far as this article is concerned, it is reasonable to ignore it.

The nature of Bode's Law

Is Bode's Law just a fluke result which happens to work for the first seven planets and the asteroid belt? Or does it encapsulate some underlying truth about the formation of our solar system, Neptune excepted?

It is important to realise that Bode's Law is not a law in the mathematical sense. It provides an approximate, but not exact, description of the positions of some of the planets. It is based only on observation, not on any theoretical considerations.

However, it is often the case that experimental observations lead to theoretical understanding. The next stage is to form a model to explain the observations. If it is a good model it can then be used to predict other results.

So for Bode's Law to be of any value, it must be supported by a theoretical model for the formation of our solar system. It is also possible that such a model could be refined to explain the anomaly of Neptune.

A single counter-example is all that is needed to disprove a mathematical law. The situation is not quite so definite for a model; it may still have some value as a partial explanation of the situation.

Alternatives to Bode's Law

Bode's Law may be written as

$$R_1 = 6$$

$$R_n = 6 + (4.5 \times 2^{n-2}) \qquad \text{for } n \geqslant 2,$$

where R_n is the distance of the nth planet from the Sun.

There are several arbitrary constants in this. The numbers 6, 4.5 and 2 have all, in effect, been picked to fit the data. The type of formula has also been chosen to match the data. There are many other types of formula that could have been tried; maybe Titius did so before finding one that worked well.

You can fit the data for any 8 planets exactly using a polynomial formula with 8 unknowns, for example

$$R_n = a + bn + cn^2 + dn^3 + en^4 + fn^5 + gn^6 + hn^7.$$

Substituting for Mercury ($n = 1$) gives

$$5.8 = a + b + c + d + e + f + g + h$$

and for Venus ($n = 2$) gives

$$10.8 = a + 2b + 4c + 8d + 16e + 32f + 64g + 128h$$

and so on. You end up with 8 equations in 8 unknowns and so an exact solution is theoretically possible.

However, you do not actually need anything like as many unknowns to produce the sort of accuracy that Bode's Law achieved. Take, for example, the polynomial with 3 unknowns

$$R_n = an + bn^3 + cn^5.$$

Substitute in the data for Venus, Jupiter and Uranus. You find:

$$a = 5.849... \qquad b = -0.1510... \qquad c = 0.009687... .$$

The predictions from this formula are given in Table 3. Those for Venus, Jupiter and Uranus are of course correct.

Planet	Prediction	Distance from the Sun ($\times 10^7$ km)
Mercury	5.7	5.8
Venus	10.8	10.8
Earth	15.8	15.0
Mars	23.7	22.8
Asteroids	40.6	
Jupiter	77.8	77.8
Saturn	152.0	142.7
Uranus	286.9	286.9

Table 3

The point is not that there is something special about the polynomial formula we have found, but that it is not difficult to find a different formula which fits the data at least as well as Bode's Law.

So Bode's Law could indeed be just a fluke.

Models for Planetary Systems

Bode's Law was a first attempt to provide information about the distribution of planets within a system. While the particular form proposed in the 18th century may prove to be of no more than historical interest, the underlying problem is one of the key issues which any theory (or model) for the evolution of a planetary system must explain.

At the moment, there are two main types of theory of how the solar system formed.

According to the *Solar Nebula Theory*, the Sun condensed from a cloud of gas. As it contracted under its own gravity, it spun faster and faster, like an ice-skater. Eventually it threw off a ring of material which then formed into the planets with nearly circular orbits. Although this theory is widely held, there are major problems associated with it. For example, it predicts that the Sun should be spinning much faster than it actually is.

In the *Capture Theory*, the planets were formed from material captured from a passing cloud of gas. This occurred at a time when both the Sun and the passing cloud were condensing from a much larger cloud of gas. Supporters of this theory claim that it can explain all the major features of the solar system. Planetary orbits would initially be highly elliptical but the effect of gravitational forces between the planets would be to make their orbits progressively more nearly circular.

The origins of the planets are completely different according to these two theories. In one, the planets were formed from material from within the Sun; in the other, the material came from elsewhere.

At the moment, we have only one reasonably complete set of data, that from our own solar system, by which we can compare different theories. However, the situation is changing. At the time of writing this article, over 60 stars have been found to have an accompanying planet. In 1999, for the first time, a star (υ *Andromeda*) was discovered to have at least three planets. All of the planets found so far are large, at least the size of Jupiter; techniques are not yet available to detect small or medium-sized planets.

However we can reasonably hope that in a few years it will be possible to find whole planetary systems. They may give evidence to support either theory – or even a totally new one.

The orbits of many of the planets that have been discovered are highly elliptical. This may be taken as evidence in favour of the Capture Theory.

For Examiner's Use

1 Justify the statement "A circle has eccentricity 0" on line 17.

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.... [2]

2 Justify the statement in the paragraph starting at line 33 that the distance of Neptune from the Sun does not agree with that predicted by extending Bode's Law.

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.... [3]

3 Bode's Law is formed using the sequence in line 22,

$$0, 1, 2, 4, \ldots, 64.$$

State the number which should replace 0 as the first term if the whole sequence is to be a geometric progression.

Find the distance of Mercury from the Sun given by this new first term.

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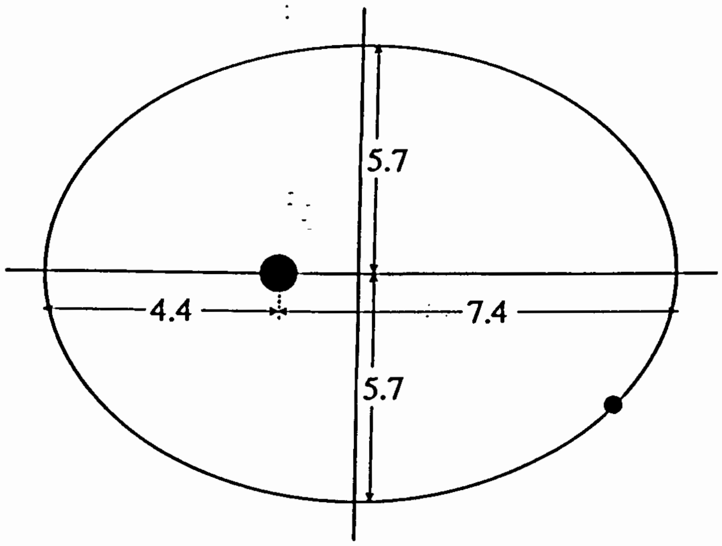
.... [3]

4 The ellipse in the diagram below illustrates the orbit of Pluto. The distances are given in units of 10^9 km.



Calculate the eccentricity of Pluto's orbit.

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.... [2]

5 In the article, distances are given in units of 10^7 km, so that the distance of Earth from the Sun is given as 15. When the law was originally put forward the metric system did not exist and the distance of Earth from the Sun was called 10 units.

Give R_1 and the formula for R_n in these old units.

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.... [2]

6 The table below gives the mean distances of the five major moons of Uranus from the planet.

n	Moon	Distance R ($\times 10^4$km)
1	Miranda	12.9
2	Ariel	19.1
3	Umbriel	26.6
4	Titania	43.6
5	Oberon	58.4

A student used three of the moons to obtain the approximate relationship

$$R = 13.66n - 0.7875n^3 + 0.02833n^5.$$

Which three moons did the student use?

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.... [2]

7 The article ends "The orbits of many of the planets that have been discovered are highly elliptical. This may be taken as evidence in favour of the Capture Theory".

Justify the final statement.

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.... [1]