C4: Comprehension question

Rationale

The aim of the comprehension question is to foster an appreciation among students that, in learning mathematics, they are acquiring skills which transcend the particular items of the specification content which have made up their course.

The objectives are that candidates should be able to:

(i) read and comprehend a mathematical argument or an example of the application of mathematics;

(ii) respond to a synoptic piece of work covering ideas permeating their whole course;

(iii) appreciate the relevance of particular techniques to real-world problems.

Description and conduct

The examination for Section C of *Applications of Advanced Mathematics (C4)* includes a comprehension question on which candidates are expected to take no more than 40 minutes. The question takes the form of a written article followed by questions designed to test how well candidates have understood it.

Candidates are allowed to bring standard English dictionaries into the examination and those for whom English is a second language are also allowed a translation dictionary. However care will be taken in preparing the question to ensure that the language is readily accessible.

Content

By its nature, the content of the written piece of mathematics cannot be specified in the detail of the rest of the specification. However knowledge of GCSE and *C1*, *C2* and *C3* will be assumed, as well as the content of the rest of this unit. Candidates are expected to be aware of ideas concerning accuracy and errors. The written piece may follow a modelling cycle and in that case candidates will be expected to recognise it. No knowledge of mechanics will be assumed.

Geodesic domes

Introduction

What do a football, a carbon 60 molecule and the Eden Project in Cornwall all have in common? Read on .... .

A football

The most familiar of these objects is a football and the pattern shown in Fig. 1 is commonly used. It is made up of regular hexagons and regular pentagons. How does it work?

Fig. 1

At every vertex, three shapes meet. Two of these are hexagons, with internal angles of 120°; the third is a pentagon with internal angles 108°. The sides of the hexagons and the pentagon are all the same length. Fig. 2 shows these three shapes meeting at a point.

When they are drawn like this on a plane surface, the sheet of paper, they do not join up. The sides OA and OD do not lie alongside each other; instead there is a gap of 12°, the angle AOD.

By contrast, Fig. 3 shows three hexagons meeting exactly at a point, without a gap.

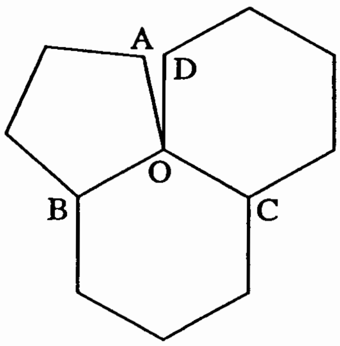

Fig. 2

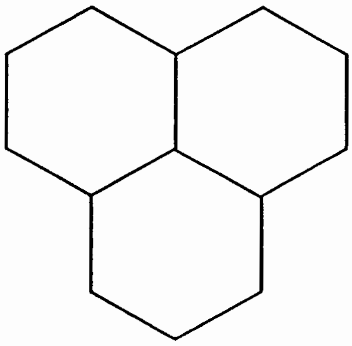

Fig. 3

If you were to cut out Fig.2 and fold along the lines OB and OC, where the shapes meet, you could then make lines OA and OD lie alongside each other. Instead of a plane figure you would then have a 3-dimensional one, either concave or convex according to how you look at it. Fig. 2 is part of the net for the football shape.

The same cannot be done with Fig. 3. It is a plane 2-dimensional figure and no folding along its edges will make it 3-dimensional.

The Platonic solids

To understand the football better, it is helpful to start by thinking about regular polyhedra. (A *polyhedron* is a solid shape; the plural of polyhedron is *polyhedra*.)

At each vertex of a polyhedron at least 3 plane faces meet.

In a regular polyhedron, each of the faces is a regular polygon and all of them are the same polygon. So the football is not a regular polyhedron because some of its faces are hexagons and others are pentagons.

What shapes are possible for the faces? It must be possible for 3 or more of them to fit together leaving a gap. Fig.3 illustrates the fact that they cannot be hexagons. The internal angle of a regular hexagon is 120° and, since $3 \times 120° = 360°$, there is no gap where they meet and so it is impossible to fold them into a 3-dimensional shape.

Regular polygons with more than 6 sides have internal angles greater than 120° and so when three or more of them meet at a point there is an overlap rather than a gap. This is shown for the heptagons in Fig. 4. They also cannot be the shapes of faces of a regular polyhedron.

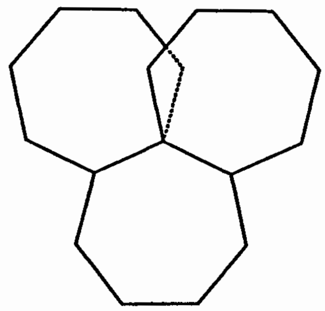

Fig. 4

Thus the only possible shapes for the faces of regular polyhedra are the regular pentagon, the square and the equilateral triangle.

The internal angle of a regular pentagon is 108° and, since

$$3 \times 108° < 360° \quad \text{and} \quad 4 \times 108° > 360°,$$

it is possible to have 3, but not 4 or more, regular pentagons meeting at a vertex.

Similarly it is possible to have 3 but not 4 squares meeting at a vertex.

In the case of equilateral triangles, 3, 4 and 5 are all possible but not 6 since $6 \times 60° = 360°$.

There are thus just five possibilities. In fact, each of these does indeed give rise to a regular polyhedron, as summarised in Table 5 and illustrated in Fig. 6. They are called the Platonic solids.

Number of sides of each face	Shape of each face	Number of faces meeting at each vertex	Name of regular polyhedron	Number of faces
5	Pentagon	3	Dodecahedron	12
4	Square	3	Cube	6
3	Triangle	3	Tetrahedron	4
3	Triangle	4	Octahedron	8
3	Triangle	5	Icosahedron	20

Table 5

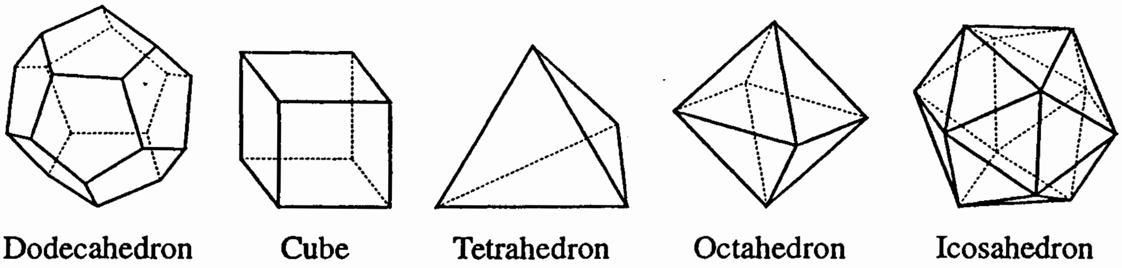

Fig. 6

Truncation

Because of symmetry, it is possible to draw spheres that pass through each of the vertices of these polyhedra. The regular polyhedron that is "closest" to its surrounding sphere is the icosahedron; it has the greatest number of faces.

By cutting off its vertices, you can make any of these polyhedra fit closer to a sphere. This process is called *truncation*; it increases the number of faces. Fig. 7 illustrates the process for a cube.

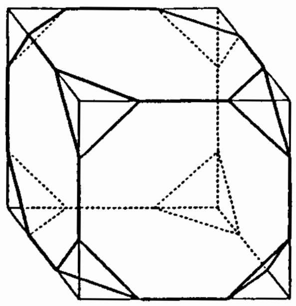

Fig. 7

The original vertices have been replaced by equilateral triangles, all of the same size. The size of the triangles is such that the faces of the original cube have changed from squares to regular octagons. The new shape has two different sorts of faces and so is not a regular polyhedron.

Now think about doing the same thing to a regular icosahedron. At each vertex, 5 equilateral triangles meet The net for one vertex and the new edges are shown in Fig. 8. If the edges AB and AC are joined, a 3-dimensional shape, either concave or convex according to how you look at it, is formed. A view of this shape is shown in Fig. 9.

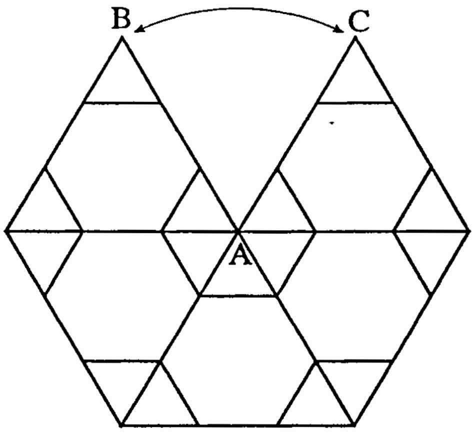

Fig. 8

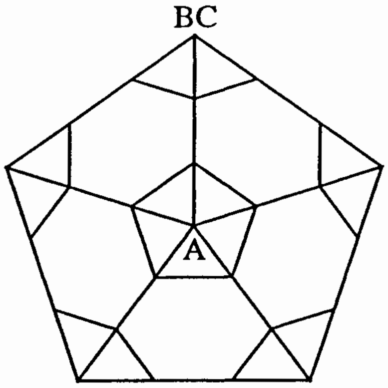

Fig. 9

When the icosahedron is truncated, each of its 12 vertices is replaced by a pentagon, and each of its 20 faces becomes a hexagon. The shape that is used for footballs is like this, with 20 hexagons and 12 pentagons. However, there is some flexibility in the material that a football is made of, so when it is inflated the faces bulge a bit and are not quite flat.

Carbon 60

The same structure occurs naturally in the carbon 60 molecule, illustrated in Fig. 10. The molecule C_{60} is one of the most stable known. The work which led to its discovery was done in the last 20 years and gained its discoverers, Kroto (then of Sussex University), Curl and Smalley, the 1996 Nobel prize for Chemistry.

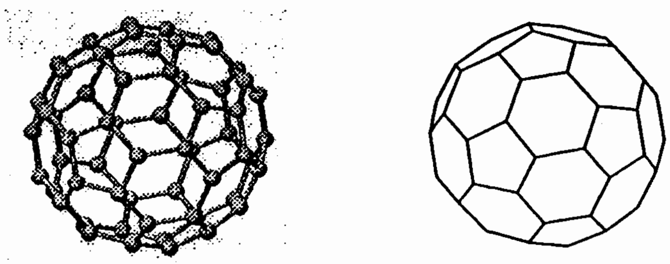

Fig. 10

Geodesic domes

The football and the C_{60} molecule are examples of a *geodesic* structure. The key feature is that all its vertices lie on a sphere. (The word "geodesic" means "following the surface of the earth".)

Geodesic structures are strong, as evidenced by the stability of the C_{60} molecule, and so are used by civil engineers in the design of domes. The edges of these domes are made of straight rigid rods and a suitable material, for example glass, is used for the faces. The design of geodesic domes was pioneered by Richard Buckminster Fuller and in his honour C_{60} has been named *buckminsterfullerine.*

There are different types of geodesic domes. One of the most commonly used is based on the same structure of pentagons and hexagons as the football. Because only three faces meet at each vertex, the number of rods used in the construction of such a dome is small, and so the structure can be relatively light.

However, engineers often create more faces by using designs based on patterns like that illustrated in Fig. 11. This new pattern has more hexagons, but the number of pentagons remains fixed at 12, one at each vertex of the original icosahedron.

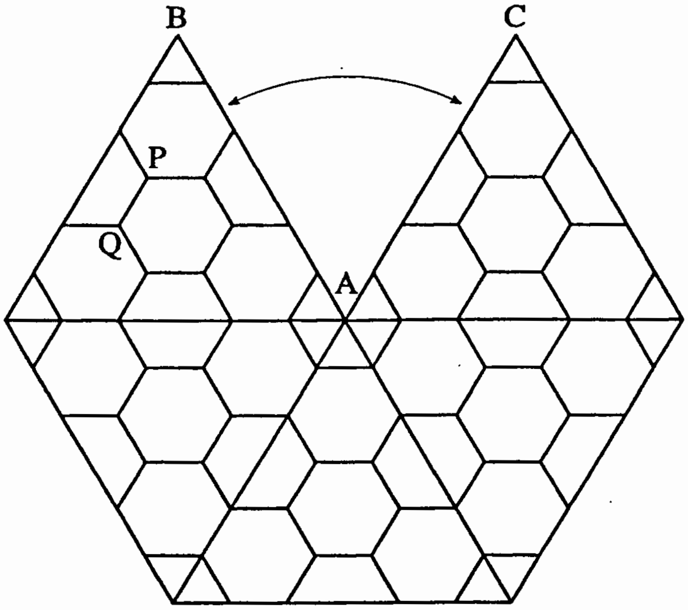

Fig. 11

Notice that all the vertices in the resulting geodesic dome must lie on a sphere. To achieve this, many of the new vertices (e.g. P and Q) have to be moved short distances outwards, each along a radius. As a consequence, some of the hexagons on the surface of the dome are not quite regular.

Patterns like that in Fig. 11 can be drawn with more (and smaller) hexagons. So a geodesic structure can be constructed with as many hexagons as required, but there will still be just 12 pentagons.

Fig. 12 shows this type of dome in use at the Eden Project in Cornwall.

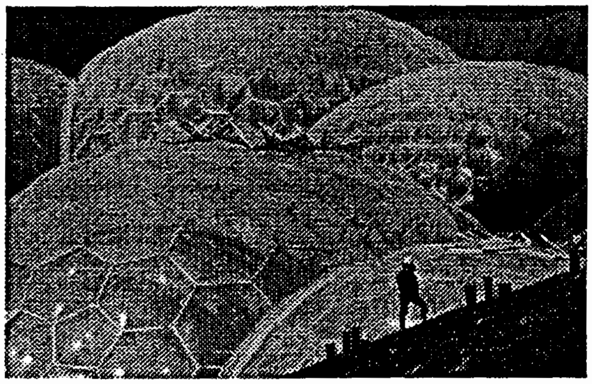

Fig. 12

For Examiner's Use

1 Explain why it is not possible to construct a regular polyhedron whose faces are regular octagons.

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2 Euler's Law states that $E = F + V - 2$, where E is the number of edges on a polyhedron, F the number of faces and V the number of vertices. Show that the regular octahedron illustrated in Fig. 6 obeys this law.

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....[2]

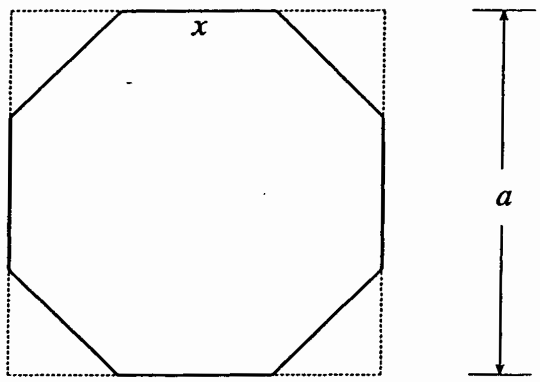
3 A distant planet is threatened with global cooling. In order to preserve heat, the inhabitants build a vast geodesic dome, of the general type described on page 6, around the planet. It consists of 2 999 990 hexagons and some pentagons. How many pentagons are there?

....[1]

For Examiner's Use

4 The diagram shows the front elevation of the truncated cube in Fig. 7. It is a regular octagon. The original cube had edges of length a. Show that the length marked x is given by

$$x = a(\sqrt{2} - 1).$$



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5 A regular tetrahedron is truncated. The new faces are all regular polygons. What shapes are these polygons and how many of each are there.

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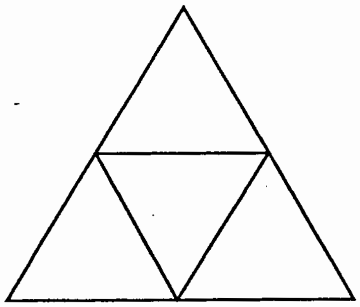
For Examiner's Use

6 A different type of geodesic structure, also based on the icosahedron, involves replacing each of the original faces by 4 smaller triangular faces, as shown below.

Find how many faces, edges and vertices the complete structure made this way would have.



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Copyright Acknowledgements:

Insert, Fig. 12

© photograph of Eden Project domes.
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