OCR
Oxford Cambridge and RSA

Friday 23 June 2017 – Morning

A2 GCE MATHEMATICS (MEI)

4754/01B Applications of Advanced Mathematics (C4) Paper B: Comprehension

QUESTION PAPER

Candidates answer on the Question Paper.

OCR supplied materials:
- Insert (inserted)
- MEI Examination Formulae and Tables (MF2)

Other materials required:
- Scientific or graphical calculator

Duration: Up to 1 hour

Candidate forename		Candidate surname	

Centre number						Candidate number				

INSTRUCTIONS TO CANDIDATES

- The Insert will be found inside this document.
- Write your name, centre number and candidate number in the boxes above. Please write clearly and in capital letters.
- Use black ink. HB pencil may be used for graphs and diagrams only.
- Answer **all** the questions.
- Read each question carefully. Make sure you know what you have to do before starting your answer.
- Write your answer to each question in the space provided. Additional paper may be used if necessary but you must clearly show your candidate number, centre number and question number(s).
- Do **not** write in the bar codes.
- The Insert contains the text for use with the questions.
- You are permitted to use a scientific or graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets **[ ]** at the end of each question or part question.
- You may find it helpful to make notes and do some calculations as you read the passage.
- You are **not** required to hand in these notes with your question paper.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.
- The total number of marks for this paper is **18**.
- This document consists of **8** pages. Any blank pages are indicated.

1 State the set of values of x_0 for which the iteration

$$x_{n+1} = 2.5x_n(1-x_n)$$

(i) converges to a single non-zero number, **[1]**

(ii) has all terms from x_1 onwards equal to zero. **[1]**

1(i)	
1(ii)	

2 **(i)** Use the algebraic method indicated in lines 68 to 70 to find the equilibrium point of the iteration

$$x_{n+1} = 1.6x_n(1-x_n).$$ **[2]**

(ii) Show that the iteration

$$x_{n+1} = x_n^2 + 2$$

does not have any points of equilibrium. **[2]**

2(i)	
2(ii)	

3 One of the assumptions for the model used for the population of squirrels in the text was that there are no predators.

An alternative model is proposed in which predators kill a fixed number of squirrels each year.

An iterative equation for this model is given by

$$x_{n+1} = kx_n(1-x_n) - 0.25.$$

In the table below x_0 is taken to be 0.55 and four different values are considered for k.

(i) Complete as many of the empty cells as you need to in order to establish the outcomes for these values of k.

(ii) Comment on what the table tells you for each of the four values of k. **[6]**

3(i)

	$x_{n+1} = kx_n(1-x_n) - 0.25$			
	$k = 2$	$k = 3$	$k = 4$	$k = 5$
x_0	0.55	0.55	0.55	0.55
x_1	0.245	0.4925	0.74	0.9875
x_2				
x_3				
x_4				
x_5				
x_6				
x_7				
x_8				
x_9				
x_{10}				
...	...	...	...	...

3(ii)	

4 **(i)** Table 3 gives the first four points of bifurcation of the iteration

$$x_{n+1} = kx_n(1-x_n).$$

Feigenbaum's Constant is 4.6692 correct to 5 significant figures. Using this value for the ratio of the interval lengths, estimate the values of k for the next two points of bifurcation. **[3]**

(ii) **(*A*)** Find, S, the sum to infinity of the geometric series

$$1+\frac{1}{4.6692}+\left(\frac{1}{4.6692}\right)^2+\left(\frac{1}{4.6692}\right)^3+\dots.$$

[2]

(*B*) Using certain figures from Table 3, a value of k is estimated to be

$$k = 3.5644 + 0.0203 \times S.$$

State what happens at this value of k. **[1]**

4(i)	

4(ii)(*A*)	
4(ii)(*B*)	

END OF QUESTION PAPER

PLEASE DO NOT WRITE ON THIS PAGE

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INSERT

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Duration: Up to 1 hour

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INSTRUCTION TO EXAMS OFFICER/INVIGILATOR

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Feigenbaum's Constant

A population model

A remote island has a population of squirrels. There are no predators so that the population is limited only by the supply of food and death from natural causes.

How would the number of squirrels be expected to change from year to year?

A simple model involves the following variables.

- Time is measured in years and the year number is denoted by n. In this article, year n is also described as 'this year' and year $n+1$ as 'next year'.
- The size of the squirrel population is given as the proportion of the maximum possible population; the population is given at the start of each year. The population in year n is denoted by x_n where $0 \leqslant x_n \leqslant 1$. Similarly, in year $n+1$ the population is x_{n+1}.
- A parameter k is a measure of the reproductivity of the squirrels, and so determines the rate of growth of their population.

The model involves the following assumptions.

- The number of squirrels next year, x_{n+1}, is jointly proportional to the number this year, x_n, and to the quantity $(1-x_n)$ which represents the food available.
- The parameter k is the constant of proportionality.

The model can be expressed by the iterative equation

$$x_{n+1} = kx_n(1-x_n).$$

This is a general model for a population in a restricted environment without predators. The animals do not have to be squirrels.

Different values of *k*

To investigate this model, it is first necessary to choose values for k and for the initial population, x_0. Table 1 gives the first ten values of x_n for four particular values of k and with $x_0 = 0.5$. The numbers given in the table have been truncated; many more figures were used in calculating them.

	$x_{n+1} = kx_n(1-x_n)$			
	$k = 0.6$	$k = 1.5$	$k = 3.3$	$k = 4.5$
x_0	0.5	0.5	0.5	0.5
x_1	0.15	0.375	0.825	1.125
x_2	0.0765	0.3515...	0.4764...	−0.6328...
x_3	0.0423...	0.3419...	0.8231...	−4.649...
x_4	0.0243...	0.3375...	0.4803...	−118.2...
x_5	0.0142...	0.3354...	0.8237...	–
x_6	0.0084...	0.3343...	0.4791...	–
x_7	0.0050...	0.3338...	0.8235...	–
x_8	0.0029...	0.3335...	0.4795...	–
x_9	0.0017...	0.3334...	0.8236...	–
x_{10}	0.0010...	0.3333...	0.4794...	–
...	...	...	...	...

Table 1

The different values of k used in Table 1 lead to four different outcomes for the population in this model.

- When $k = 0.6$, $x_n \to 0$. Eventually the population dies out.
- When $k = 1.5$, $x_n \to 0.3333... = \frac{1}{3}$. The population attains a stable equilibrium level.
- When $k = 3.3$, the population alternates between a high value of 0.823... and a low value of 0.479... in successive years.
- When $k = 4.5$, the population appears to become negative and so to have died out. In the first year the population exceeds the limit of 1 and so the model has broken down.

In each case in Table 1 the value of x_0 was taken to be 0.5, the middle of the possible values of x_0. If you try other starting values you will find that the final outcomes are the same for any values of x_0 between, but not including, 0 and 1.

Overall the model suggests that having too few or too many young can both be fatal for the population.

Possible outcomes

This iteration can be used as a population model, but it can also be thought of as a mathematical iteration in its own right with an interesting variety of possible outcomes.

At this stage it is helpful to extend the notation used to include the letter x. This denotes the value, or values, of x_n as n tends to infinity.

In Fig. 2, these limiting values, x, are plotted for values of k between 0 and 3.6. For larger values of k there are no limiting values. It is assumed that $0 < x_0 < 1$.

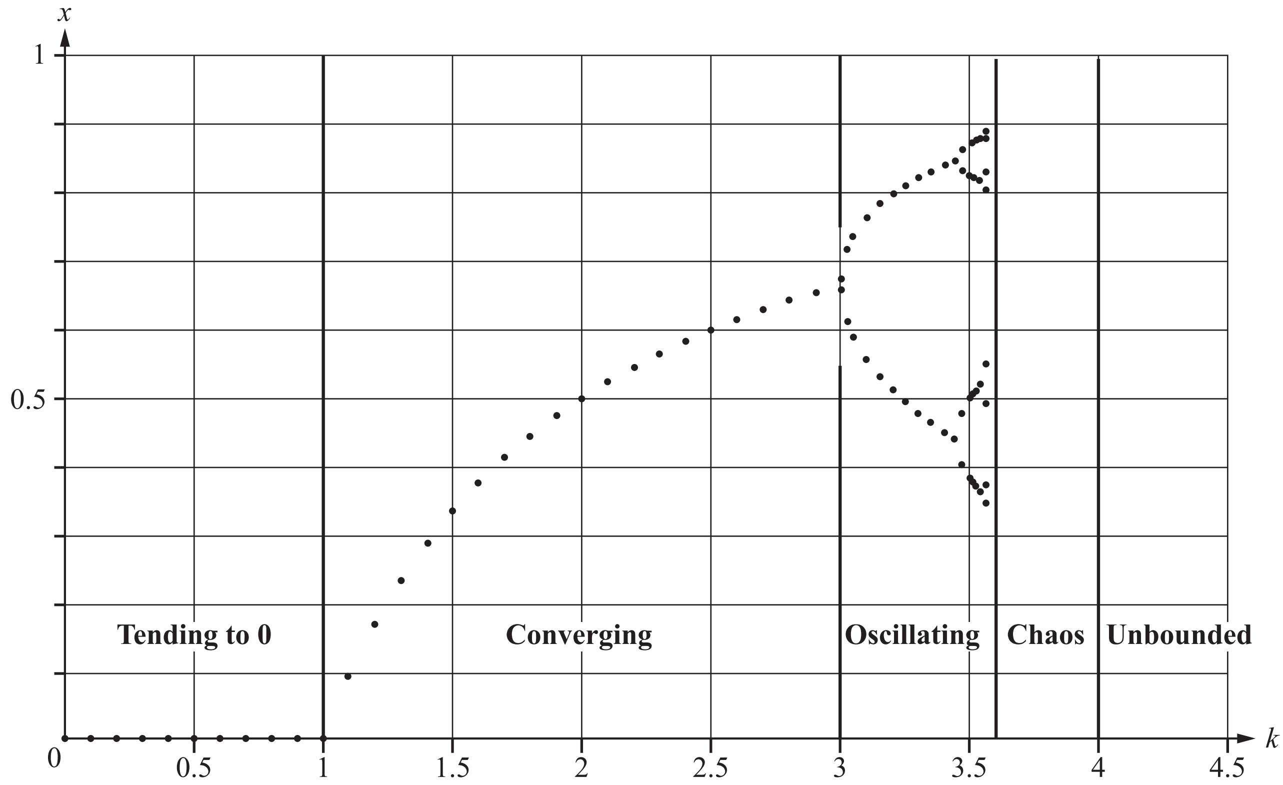

Fig. 2

Fig. 2 shows that, for this iteration, five different types of outcome are possible according to the value of k. These are described below.

Tending to zero

For $0 \leqslant k \leqslant 1$, the iteration converges to zero.

Converging to a single non-zero value

For $1 < k < 3$, the iteration converges to a non-zero value between 0 and 1.

An example of this is given in Table 1 for $k = 1.5$. The iteration is found to converge to $\frac{1}{3}$. This value may be described as an *equilibrium point*.

It can be found algebraically. Denoting it by x gives the equation

$$x = 1.5x(1-x)$$

which has a solution of $x = 0$ or $\frac{1}{3}$.

Oscillating

Table 1 shows that for $k = 3.3$, x_n oscillates between two values. There is a range of values of k for which this occurs.

- For $k = 2.99$, x_n converges slowly to 0.6655... . At $k = 3$, it starts to oscillate. After 5000 iterations the low value is 0.6633... and the high value is 0.6699... . This is a cycle of length 2.

Thus it is found, for example by experiment using a spreadsheet, that the smallest value of k for which the iteration oscillates is 3. Such a value of k where the iteration splits is called a *point of bifurcation*.

However, $k = 3$ is not the only point of bifurcation. At $k = 3.449...$ there is a further point of bifurcation at which the cycle of length 2 becomes a cycle of length 4.

- For $k = 3.5$, the four values of x are 0.3828..., 0.8269..., 0.5008... and 0.8749... .

Another point of bifurcation occurs at $k = 3.544...$. At this point the length of the cycle goes up to 8. Further points of bifurcation give cycles of length 16, then 32, then 64 and so on.

Chaos

The pattern of cycles of increasing length does not continue indefinitely as k increases. For larger values of k, for example $k = 3.8$, the outcomes have no pattern, and so the situation is described as *chaos*. It is very difficult to distinguish between chaos and a long cycle; a sophisticated computer program is required.

A feature of chaos is that the iteration remains bounded. The values of x_n are always between 0 and 1.

Unbounded outcomes

For values of $k \geqslant 4$, the outcomes cease to be bounded. An example of this occurs in the final column of Table 1, where the value of k is 4.5.

Feigenbaum's Constant

To summarise, from $k = 1$ to 3 the iteration $x_{n+1} = kx_n(1 - x_n)$ converges to a single non-zero value. There is then a point of bifurcation and this is followed by further points of bifurcation. It is evident from Fig. 2 that the intervals between successive points of bifurcation become progressively shorter.

Information about these intervals is given in Table 3. The numbers in this table have been rounded to 4 decimal places.

Cycle length	**Boundary values of *k***		**Interval**	$\dfrac{\textbf{Previous interval}}{\textbf{This interval}}$
1	1	3	2	–
2	3	3.4495	0.4495	4.4494
4	3.4495	3.5441	0.0946	4.7516
8	3.5441	3.5644	0.0203	4.6601
16	3.5644	...	...	...

Table 3

The right hand column gives the ratio by which the length of this interval decreases with successive cycles. The three values of this ratio in Table 3 are close together.

In the 1970s, Mitchell Feigenbaum started to investigate this ratio. He discovered that similar patterns of bifurcation are found with many other iterations; examples include $x_{n+1} = x_n^2 + k$ and $x_{n+1} = k\sin(\pi x_n)$.

He also discovered that the ratio tends to a definite limit and that this has the same value for all iterations that show this pattern of bifurcation.

His work was conducted to a very high level of accuracy and covered many more cycles than the small number considered here. The limited power of computers in those days meant that it was an enormous undertaking.

Feigenbaum was immediately convinced of the importance of his discovery.

'I called my parents that evening and told them I had discovered something truly remarkable, that, when I had understood it, would make me a famous man.'

He is indeed now a famous man and the ratio he discovered is called Feigenbaum's Constant. It has now been found to over 1000 figures; the first ten of these are 4.669 201 609.

Question		Answer	Marks	Guidance
1.	(i)	(All x_0 for which) $0 < x_0 < 1$	B1	Condone as separate inequalities – allow x, x_n, etc. for x_0 Accept (0, 1) (i.e. correct set-notation) – must be strict inequalities
			[1]	
	(ii)	$x_0 = 0$ or 1	B1	Both required - allow x, x_n, etc. for x_0 - condone 0, 1 stated without x_0 (0 and 1 must not appear as part of a range of values)
			[1]	

2.	**(i)**	$x=1.6x(1-x)$ $x(1.6x-0.6)=0$ $(x=0 \text{ or})\ x=\dfrac{0.6}{1.6}=0.375 \text{ or } \dfrac{3}{8}$	**M1** **A1**	For correctly replacing x_{n+1} **and** x_n with x - must be using 1.6 Using an iterative approach or stating the correct answer with no working - M0 A0 Accept as a minimum the correct equation stated in terms of x only followed by the correct answer for both marks
			[2]	
2.	**(ii)**	$x^2-x+2=0$ Discriminant $=(-1)^2-4\times1\times2=-7$ or $x=\dfrac{1\pm\sqrt{(-1)^2-4(1)(2)}}{2}$ so no (real) roots (oe)	**M1** **A1**	Re-arranging **and** correct attempt at either the discriminant or solving using the correct formula/completing the square for their 3-term quadratic equation. Allow consideration of $\sqrt{b^2-4ac}$ with correct substitution of their values only Correct working with a correct conclusion (so if discriminant stated must be -7 but may be awarded for $1-4(2)\Rightarrow$ no (real) roots/no points of equilibrium) No marks for non-algebraic approach
			[2]	

PLEASE NOTE THAT THE MARKS FOR BOTH 3(i) and 3(ii) MUST BE AWARDED TOGETHER (MAXIMUM OF 6)

3. **(i)**

$$x_{n+1} = kx_n(1-x_n) - 0.25$$

	k = 2	k = 3	k = 4	k = 5
x_0	0.55	0.55	0.55	0.55
x_1	0.245	0.4925	0.74	0.9875
x_2	0.1199...	0.4998...	0.5196	-0.1882...
x_3	-0.0388...	0.4999...	0.7484...	...
x_4	...	0.5	0.5030...	...
x_5	...	...	0.7499...	...
x_6	...	...	0.5000...	...
x_7	...	...	0.7499...	...
x_8	...	...	0.5000...	...
x_9			0.75	
x_{10}			0.5	
...	...	...	...	...

There may be some variation in the columns for k = 3 and k = 4. Some calculators will round to the final values earlier than others. **In general answers must be given to at least 2dp**

Allow correct truncated or rounded answers

One mark for each column taken far enough

B1 $k = 2$ correct to at least x_3

B1 $k = 3$ correct to at least x_3 (accept 0.5)

B1 $k = 4$ correct to at least x_6 (accept 0.5)

B1 $k = 5$ correct to at least x_2

isw after reaching the values for each column as stated above

3.	**(ii)**	For k = 2 it is **unbounded** or '**dying out**' For k = 3 it **converges** (to x = 0.5) or **stable** or **equilibrium** For k = 4 it **oscillates** (between x = 0.5 and x = 0.75) or **alternates** but not **fluctuates** For k = 5, it is **unbounded** or '**dying out**'	**B2**	All 4 correct Allow B1 for 2 correct Allow equivalent to 'dying out' provided unambiguous Must be one of these three words Must be one of these two words Allow equivalent to 'dying out' provided unambiguous
			[6]	

4.	**(i)**	$3.5644+\frac{0.0203}{4.6692}$	**B1**	Correct method - allow a slip in one value (e.g. 4.692) only – either correct value for *k* implies this mark
		$=3.5687$	**B1**	Accept 3.569 (3.5687476…) – no ft from incorrect values – accept if given as the upper bound of an interval
		$3.5687+0.0203\times\left(\frac{1}{4.6692}\right)^2=3.5696$ to 5 sf	**B1**	Accept 3.5697 or 3.570 but not 3.57 – no ft from incorrect values – accept if given as an upper bound
			[3]	
4	**(ii)(*A*)**	$S=\frac{1}{1-\frac{1}{4.6692}}$	**M1**	Use of the correct formula for the sum to infinity of a GP – allow a slip in the value of *r* only (e.g. 4.692) – allow *r* stated to 2dp or better (r = 0.21416945…)
		$S=1.2725...$	**A1**	Accept 1.273 not 1.27 (1.2725389…) M0 A0 if formula not used
			[2]	
4	**(ii)(*B*)**	It is an estimate of a value of *k* for which chaos is occurring	**B1**	Accept any mention of ‘chaos’
			[1]	