MEI NUMERICAL METHODS JUNE 2007 MODEL SOLUTIONS

Q1

Remember bisection method. Looks at boundary given by change of sign, find the midpoint, and find sign of f(x). It will take the place of the boundary with same sign for f(x).

i)

$f(x)=0 \to x^2+\sqrt{1+x}-3=0$

$f(1)=1+\sqrt{2}-3=\sqrt{2}-2<0$ Hence change of sign, root lies between 1 &1.4

$f(1.4)=1.4^2+\sqrt{2.4}-3=0.509>0$

We now look at midpoint 1.2

$f(1.2)=1.2^2+\sqrt{2.2}-3=-0.0767<0$ root lies in {1.2,1.4} with max error 0.1

We now look at midpoint 1.3

$f(1.3)=1.3^2+\sqrt{2.3}-3=0.2065>0$ root lies in {1.2,1.3} with max error 0.05

We now look at midpoint 1.25

$f(1.25)=1.25^2+\sqrt{2.25}-3=0.0625>0$ root lies in {1.2,1.25} with max error 0.025

Midpoint 1.225

Hence we have an approximation to the root of 1.225 with max error of 0.025

Note max error reduces by factor 2 each time. So next will give 0.0125, then 0.00625 and then 0.003125. Hence we need 3 more iteration to achieve less than 0.005 max error.

Q2 We have to find $\int_0^{.5} \frac{1}{1+x^4}\mathrm{d}x$

$$M_1 = h[y_{m1}] = h\left[f(0.25)\right] = 0.5 * \frac{1}{1+0.25^4} \Rightarrow \boxed{M_1 = 0.498054}$$

and

$$T_1 = \frac{h}{2}[y_0+y_1] = \frac{0.5}{2}\left[f(0)+f(0.5)\right] = 0.25*\left(\frac{1}{1+0^4}+\frac{1}{1+0.5^4}\right) \Rightarrow \boxed{T_1 = 0.485294}$$

We know

$$S_n = \frac{2M_n+T_n}{3} \Rightarrow S_1 = \frac{2M_1+T_1}{3} = \frac{2*0.498054+0.485294}{3} \Rightarrow \boxed{S_1 = 0.493801}$$

For extrapolation use

$$S_\infty = S_{2n} + \frac{S_{2n}-S_n}{15} \Rightarrow S_\infty = 0.493952 + \frac{(0.493952-0.493801)}{15} \Rightarrow \boxed{S_\infty = 0.493962}$$

Note the have not used the full expansion so different answer.

Q3

Cosine rule

$$a^2 = 3^2 + 4^2 - 2*4*3\cos(90+\varepsilon) \Rightarrow a = \sqrt{25-24\cos 95} \Rightarrow \boxed{a = 5.204972}$$

This is the exact answer. If we approximate $\cos(90+\varepsilon) = -\frac{\pi\varepsilon}{180} = -\frac{\pi 5}{180} = -\frac{\pi}{36}$

then the approximation A is given by $A = \sqrt{25 - 24\frac{-\pi}{36}} \Rightarrow \boxed{A = 5.205228}$

Absolute Error $= |a - A| = |5.204972 - 5.205228| = \boxed{0.000255}$

Relative Error is $\frac{|a-A|}{a} = \boxed{0.000049}$

Q4

As $X = x(1+r) \Rightarrow r = \frac{X}{x} - 1 = \frac{X - x}{x}$ r is the relative error.

X^n is approx to $x^n \Rightarrow X^n \approx x^n(1+r)^n$ Now for the last term consider the binomial expansion of the bracket. Remember the first terms are always of the form $1+nr+$ something with r^2+.. We are asked to keep only 2 first terms so $X^n \approx x^n(1+nr)$

And the relative error is $\frac{X^n}{x^n} \approx (1+nr) \Rightarrow \boxed{\frac{X^n - x^n}{x^n} \approx nr}$

Relative Error in approx to $\pi = \frac{22}{7}$ is given by $\frac{\frac{22}{7} - \pi}{\pi} = 4.025\times10^{-4}$

Hence approx relative error in

π^2 is 2 times error in $\frac{22}{7} = 2\times4.025\times10^{-4} = \boxed{8.050\times10^{-4}}$ and in

$\sqrt{\pi}$ is 0.5 times error in $\frac{22}{7} = 0.5\times4.025\times10^{-4} = \boxed{2.013\times10^{-4}}$

Q5

Lagrange's formula give for three points

$$f(x) = \frac{(x-x_2)(x-x_3)}{(x_2-x_1)(x_3-x_1)}f(x_1) + \frac{(x-x_1)(x-x_3)}{(x_1-x_2)(x_3-x_2)}f(x_2) + \frac{(x-x_2)(x-x_1)}{(x_2-x_3)(x_1-x_3)}f(x_3)$$

$$f(x) = \frac{(x-0)(x-4)}{(1\ \ )(5)}3 + \frac{(x-4)(x+1)}{(-1)(4)}2 + \frac{(x-0)(x+1)}{(-4)(-5)}9 = \frac{3}{5}(x)(x-4) - \frac{1}{2}(x-4)(x-1) + \frac{9}{20}x(x+1)$$

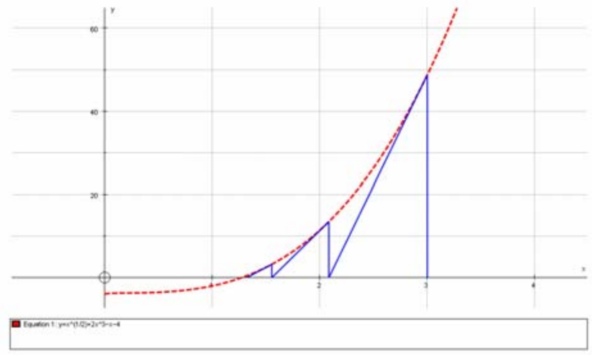
$$f(x) = 0.55x^2 - 0.45x + 2$$

and

$$f'(x) = 1.1x - 0.45$$

Hence min is at x=0.41

Q6



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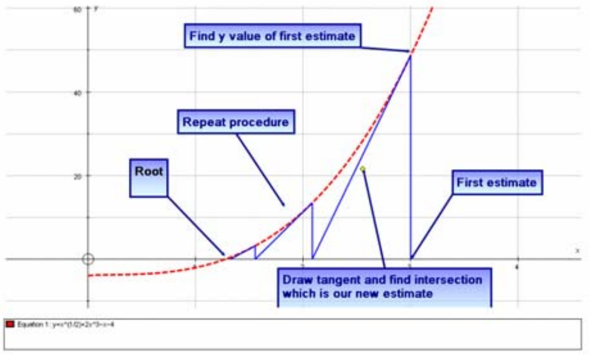
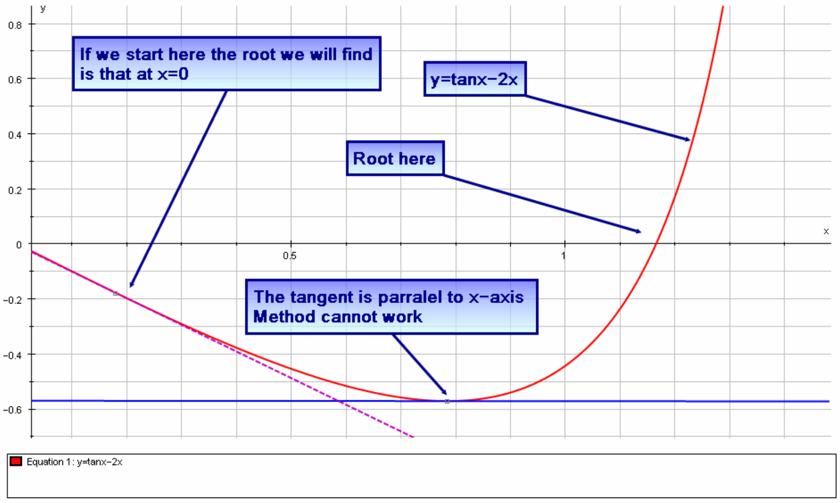


GOOD EXPLANATIONS

You need to indicated **clearly** your root, initial estimate and intersection of tangent on your graph.

For graph use TABLE mode in calculator between 0 and 1.5 and label diagram!
Note the root you want to find is non-zero, i.e. between1 and 1.2



Newton-Raphson method

The formula is $x_{r+1} = x_r - \dfrac{f(x_r)}{f'(x_r)} = x_r - \dfrac{\tan x_r - 2x_r}{\tan^2 x_r - 1}$ with the derivative given in text. Use 'program' with A,B &C in calculator. After 4 iterations we get

i	xk
0	1.2
1	1.169346024
2	1.165609311
3	1.165561193
4	1.165561185

Note the last 2 agree to 4 d.p. i.e root =1.1656 Note if first order convergence then we ratios of difference must be deceasing very fast.

x1.2	Diff	ratios
	0.034438815	
1.169346024	0.003784839	0.1218998
1.165609311	4.81259E-05	0.0128771
1.165561193	7.83367E-09	0.0001628

As the ratios decrease very rapidly we must have 1st order convergence.

Q6

You need to establish the difference table

x	G(x)	Δx	Δ^2x
1	2.87		
2	4.73	1.86	
3	6.23	1.5	-0.36
4	7.36	1.13	-0.37
5	8.05	0.69	-0.44

We can see that the second differences are not sufficiently constant for the points to be fitted by a quadratic.

x	G(x)	Δx	Δ^2x
1	2.87		
3	6.23	3.36	
5	8.05	1.82	-1.54

The approximation to g(x) using this table is

$$Q(x) = g_0 + \frac{(x-x_0)}{h}\Delta g_0 + \frac{(x-x_0)(x-x_1)}{2h^2}\Delta^2 g_0$$

$$= 2.87 + \frac{(x-1)}{2}\times 3.36 + \frac{(x-1)(x-3)}{8}\times -1.54$$

$$= \boxed{0.6125 + 0.245x - 0.1925x^2}$$

The error and relative errors are

x	Q(x)	g(x)	error	relative error
2	4.7425	4.73	**0.0125**	**0.002643**
4	7.3325	7.36	**-0.0275**	**-0.00374**