MEI M3 JANUARY 2009 WORKED SOLUTIONS

Q1 i) dimensions of force $[F]=\frac{[\mathrm{M}]\;[\mathrm{L}]}{[\mathrm{T}]^2}$ and of density $[\rho]=\frac{[\mathrm{M}]}{[\mathrm{L}]^3}$

ii) The viscosity η is defined by $F=\frac{\eta A(v_2-v_1)}{d}$ where
F is a force, d is a distance, dimension [L] and
ν is a velocity, dimension $[\upsilon]=\frac{[\mathrm{L}]}{[\mathrm{T}]}$ and Area A has $[\mathrm{L}]^2$.

Hence we have $\frac{[\mathrm{M}]\;[\mathrm{L}]}{[\mathrm{T}]^2}=\frac{[\eta][\mathrm{L}]^2}{[\mathrm{L}]}\frac{[\mathrm{L}]}{[\mathrm{T}]}=\frac{[\eta][\mathrm{L}]^2}{[\mathrm{T}]}$ and for both sides to be dimensionally consistent we need $\boxed{[\eta]=[M][\mathrm{L}]^{-1}[\mathrm{T}]^{-1}}$

iii)

We are given $v=\frac{2a^2\rho g}{9\eta}$ where a is radius, [L]. Hence we require that RHS has dimensions of $[\mathrm{L}][\mathrm{T}]^{-1}$. $[\mathrm{RHS}]=\frac{[\mathrm{L}]^2[\mathrm{M}][\mathrm{L}]^{-3}[\mathrm{L}][\mathrm{T}]^{-2}}{[\mathrm{M}][\mathrm{L}]^{-1}[\mathrm{T}]^{-1}}=\frac{[\mathrm{T}]^{-1}}{[\mathrm{L}]^{-1}}=[\mathrm{L}][\mathrm{T}]^{-1}$ as required

iv) If $R=\rho w^{\alpha}v^{\beta}\eta^{\gamma}$, is dimensionless then we need $[\rho][w]^{\alpha}[v]^{\beta}[\eta]^{\gamma}$ to be also dimensionless, with w a length [L].

Now

$$[\rho][w]^{\alpha}[v]^{\beta}[\eta]^{\gamma}=[\mathrm{M}][\mathrm{L}]^{-3}[\mathrm{L}]^{\alpha}\left([\mathrm{L}][\mathrm{T}]^{-1}\right)^{\beta}\left([\mathrm{M}][\mathrm{L}]^{-1}[\mathrm{T}]^{-1}\right)^{\gamma}$$
$$=[\mathrm{M}]^{\gamma+1}[\mathrm{L}]^{-3+\alpha+\beta-\gamma}[\mathrm{T}]^{-\beta-\gamma}$$

Hence we require $\begin{aligned}\gamma+1&=0\\-3+\alpha+\beta-\gamma&=0\\-\beta-\gamma&=0\end{aligned}$ which gives $\begin{aligned}\gamma&=-1\\\beta&=1\\\alpha&=1\end{aligned}$

v)
Calculate the Reynolds number for real case gives $R=9.375\text{x}10^7$

Use this value with new input keeping ν the velocity as unknown. Gives ν=260 m/s

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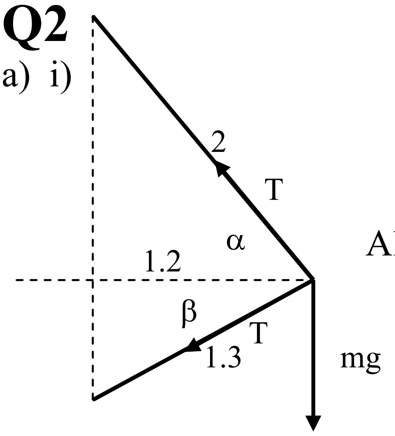
Q2

a) i)



Note: Tension is the same everywhere in the string.

Also from diagram $\cos\alpha = \frac{1.2}{2} = \frac{3}{5}$, $\cos\beta = \frac{1.2}{1.3} = \frac{12}{13}$ Hence $\sin\alpha = \sqrt{1-\left(\frac{3}{5}\right)^2} = \frac{4}{5}$, $\sin\beta = \sqrt{1-\left(\frac{12}{13}\right)^2} = \frac{5}{13}$

avoid using calculator!

<u>Equations of motions</u> Radial direction $-T\cos\alpha - T\cos\beta = -ma_r$ (1)

Vertical direction $T\sin\alpha - T\sin\beta - mg = 0$ (2)

i) From (2) we find $T = \dfrac{mg}{(\sin\alpha - \sin\beta)} = 6.37\text{N}$

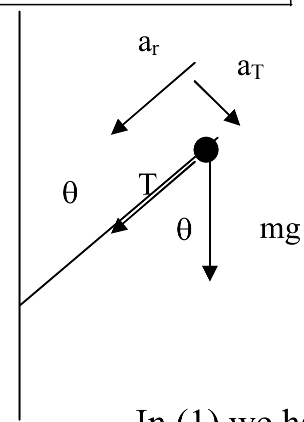
From in (1) $\dfrac{T(\cos\alpha + \cos\beta)}{m} = a_r$ and using $a_r = \dfrac{v^2}{r}$ we get

$$v = \sqrt{\frac{a_r}{r}} = \sqrt{\frac{\frac{T(\cos\alpha+\cos\beta)}{m}}{r}} = \sqrt{\frac{rT(\cos\alpha+\cos\beta)}{m}} = 6.57\text{m/s}$$

b) i)At highest position we require that $mg = ma_r$, hence as a $a_r = \dfrac{v^2}{r}$ we get

$$\boxed{v = \sqrt{rg} = \sqrt{12.25} = 3.5m/s}$$

ii)



Eq of motion

Radial $T + mg\cos 60 = ma_r$ (1)

Transverse $mg\sin 60 = ma_T$ (2)

From (2) we get $g\sin 60 = a_T \Rightarrow \boxed{a_T = 8.49\ \ \text{ms}^{-2}}$

In (1) we have two unknowns so we need to obtains extra information.

We know that E is conserved, i.e. $\Delta KE + \Delta PE = 0$, between the positions at the top θ=0 and at θ=60^0.

$\Delta KE = \frac{1}{2}mv^2 - \frac{1}{2}m3.5^2$ and $\Delta PE = mgr\cos 60 - mgr = mgr(\cos 60 - 1) = -0.5mgr$

$$\Delta KE + \Delta PE = \frac{1}{2}mv^2 - \frac{1}{2}m3.5^2 - 0.5mgr = 0$$

So and as $a_r = \dfrac{v^2}{r} \Rightarrow a_r = \dfrac{gr + 3.5^2}{r} \Rightarrow \boxed{a_r = 19.6}$

$$v^2 = gr + 3.5^2$$

We can now put a_r into (1) to get <u>T= 2.94 N</u>

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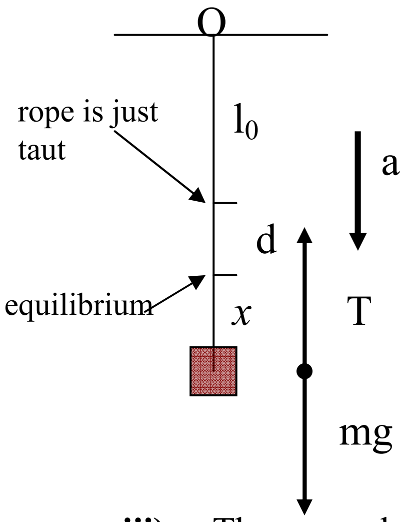
Q3



We know $l_0 = 25m$ $\quad k = \dfrac{\lambda}{l_0} = \dfrac{980}{25} = 38.2Nm^{-1}$

i) Now in eq. we have $F = kd = mg \Rightarrow d = \dfrac{mg}{k} = 1.25$

$T - mg = -m\dfrac{d^2x}{dt^2}$ $\quad$ and as $T = k(d+x)$

ii) $k(d+x) - mg = -m\dfrac{d^2x}{dt^2}$ $\quad$ and as $kd = mg$

$$-\frac{kx}{m} = \frac{d^2x}{dt^2} \Rightarrow -\frac{38.2x}{5} = \frac{d^2x}{dt^2} \Rightarrow \boxed{-7.84x = \frac{d^2x}{dt^2}}$$

iii) The general solution for SHO is $x = A\cos(\omega t) + B\sin(\omega t)$ (1) with
$\omega^2 = 7.84 \Rightarrow \boxed{\omega = 2.8}$ and the velocity is $v = -A\omega\sin(\omega t) + B\omega\cos(\omega t)$ (2)

The initial conditions are $t = 0$ $\quad x = -1.25m$ $\quad v = 8.4m/s$
into (1) & (2) gives *A=-1.25,B=3*

Hence $\underline{x = -1.25\cos(2.8t) + 3\sin(2.8t)}$ and $\underline{v = 1.25\times 2.8\sin(2.8t) + 3\times 2.8\cos(2.8t)}$

and using compound angle formula $R = \sqrt{a^2+b^2}$ and $\phi = \tan^{-1}(b/a)$
e.g for v $\quad R = \sqrt{(-1.25)^2 + (3)^2} = 3.25$ and $\phi = \tan^{-1}(3/1.25) = 1.176$

gives $\underline{x = -3.25\cos(2.8t + 1.176)}$ and differentiate for $\underline{v = 9.1\sin(2.8t + 1.176)}$

Hence max x and v are when the cos terms are equal to one, i.e. the amplitudes.
$\underline{x_{\max} = -3.25m}$ $\quad$ iv) $\underline{v_{\max} = 9.1m/s}$

v) When the rope becomes taut we know that v=8.4 m/s, given in question, and t=0

When the mass is at rest need to solve

$$0 = 9.1\sin(2.8t + 1.176) \Rightarrow \sin^{-1}(0) = 0, \pi, ...$$
$$\sin^{-1}(0) = 0 \Rightarrow 0 = 2.8t + 1.176 \Rightarrow t = -0.42$$

$$\sin^{-1}(0) = \pi \Rightarrow t = \frac{\pi - 1.176}{2.8} = 0.702$$

we need first time it gets to zero, i.e first positive value. *t=0.702*

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Q4

a) $A\bar{y} = \int_{-a}^{a} \frac{y^2}{2} dx = \int_{-a}^{a} \frac{(a^2 - x^2)}{2} dx = \frac{1}{2}\left[a^2 x - \frac{x^3}{3}\right]_{-a}^{a} = \frac{1}{2}\left[\left(a^3 - \frac{a^3}{3}\right) - \left(-a^3 - \frac{-a^3}{3}\right)\right] = \frac{1}{2}\left(2a^3 - \frac{2a^3}{3}\right) = \frac{2a^2}{3}$

and as $A = \frac{1}{2}\pi a^2$ we have $\bar{y} = \dfrac{\frac{2}{3}a^3}{\frac{1}{2}\pi a^2} = \boxed{\dfrac{4a}{3\pi}}$.

b) i)First we need volume $V = \pi\int_0^h y^2 dx = \pi\int_0^h (mx)^2 dx = \pi m^2\left[\frac{x^3}{3}\right]_0^h = \underline{\underline{\frac{\pi m^2 h^3}{3}}}$

We now need to evaluate $\pi\int_0^h xy^2 dx = \pi\int_0^h x(mx)^2 dx = \pi m^2\left[\frac{x^4}{4}\right]_0^h = \underline{\underline{\frac{\pi m^2 h^4}{4}}}$ and now we can find

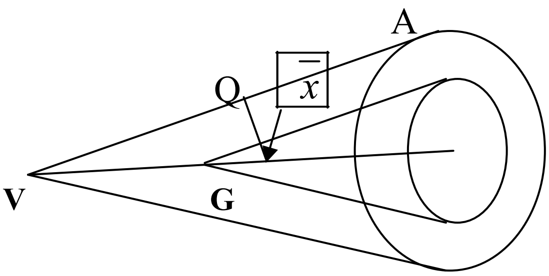
$$\bar{x} = \frac{\pi\int_0^h xy^2 dx =}{\pi\int_0^h y^2 dx} = \frac{\frac{\pi m^2 h^4}{4}}{\frac{\pi m^2 h^3}{3}} = \frac{3h}{4}$$

NOTE HOW THIS INDEPENDENT OF m

ii) Now using the result of part i) We refer to large cone as 1, small cone as 2

$\bar{x}_2 = \frac{3h_2}{4} = 0.825$ but this is from top G of small cone (h=1.1), which is at 1.3 (2.4-1.1) from top V of large cone. So if we choose top of large cone (h=2.4) as our zero

$\bar{x}_2 = 1.3 + 0.825 = 2.125$ and $\bar{x}_1 = 1.8$



Using $M_2\bar{x}_2 + M\bar{x} = M_1\bar{x}_1$

with $M = M_1 - M_2$, i.e. mass cut = mass of large cone minus mass of small one.

Now the cone are uniform, hence mass is proportional to volume and we can use these instead.

So that $\bar{x} = \dfrac{M_1\bar{x}_1 - M_2\bar{x}_2}{M} = \dfrac{V_1\bar{x}_1 - V_2\bar{x}_2}{V_1 - V_2}$. The volumes are given by $V = \frac{\pi r^2 h}{3}$, hence

$$\bar{x} = \frac{\frac{\pi r_1^2 h_1}{3}\bar{x}_1 - \frac{\pi r_2^2 h_2}{3}\bar{x}_2}{\frac{\pi r_1^2 h_1}{3} - \frac{\pi r_2^2 h_2}{3}} = \frac{r_1^2 h_1\bar{x}_1 - r_2^2 h_2\bar{x}_2}{r_1^2 h_1 - r_2^2 h_2} = \frac{0.7^2\times 2.4\times 1.8 - 0.4^2\times 1.1\times 2.125}{0.7^2\times 2.4 - 0.4^2\times 1.1} = \boxed{1.74m}$$

iii) If the cone hangs in equilibrium form Q the line $Q\bar{x}$ is perpendicular to VA.

The angle $\hat{Q}V\bar{x}$ is given by $\tan^{-1}\left(\dfrac{\text{radius of large cone}}{\text{height of large cone}}\right) = \tan^{-1}\left(\dfrac{0.7}{2.4}\right) = 0.2838$

And $\boxed{VQ = \bar{x}\cos(0.2828) = 1.67}$