

A surface has equation  $z = 3x^2 - 12xy + 2y^3 + 60$ .

- (i) Show that the point A (8, 4, -4) is a stationary point on the surface. Find the coordinates of the other stationary point, B, on this surface. [5]

$$z = 3x^2 - 12xy + 2y^3 + 60$$

$$\frac{\partial z}{\partial x} = 6x - 12y, \quad \frac{\partial z}{\partial y} = -12x + 6y^2$$

$$\text{At stat. pts, } \frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} = 0$$

$$\Rightarrow 6x - 12y = 0$$

$$6y^2 - 12x = 0$$

$$x = 2y$$

$$y^2 = 2x$$

$$\frac{y^2}{z} = 2y$$

$$\frac{y^2}{z} = x$$

$$\frac{y^2}{z} = 4y$$

subst.

$$y(y-4) = 0$$

$$y = 0 \quad \text{or} \quad y = 4$$

$$x = 2(0)$$

$$x = 2(4)$$

$$= 0$$

$$= 8$$

$$z = 0 + 60$$

$$z = 3(8)^2 - 12(8)(4) + 2(4)^3 + 60$$

$$= 64$$

$\therefore$  stat. pts at B(0, 0, 60) and A(8, 4, -4)

(ii) A point P with coordinates  $(8+h, 4+k, p)$  lies on the surface.

(A) Show that  $p = -4 + 3(h-2k)^2 + 2k^2(6+k)$ .

[3] \_\_\_\_\_

(B) Deduce that the stationary point A is a local minimum.

[3] \_\_\_\_\_

(C) By considering sections of the surface near to B in each of the planes  $x=0$  and  $y=0$ , investigate the nature of the stationary point B.

[4] \_\_\_\_\_

(A)  $z = 3x^2 - 12xy + 2y^3 + 60 \quad \text{Subst. } (8+h, 4+k, p)$

$$p = 3(8+h)^2 - 12(8+h)(4+k) + 2(4+k)^3 + 60$$

$$= 3(64 + 16h + h^2) - 12(32 + 8k + 4h + kh) + 2(4^3 + 48k + 12k^2 + k^3) + 60$$

$$= 192 + 48h + 3h^2 - (384 + 96k + 48h + 12kh) + 128 + 96k + 24k^2 + 2k^3 + 60$$

$$= -4 + 3h^2 - 12kh + 24k^2 + 2k^3$$

$$= -4 + 3(h^2 - 4kh) + 24k^2 + 2k^3$$

$$= -4 + 3((h-2k)^2 - 4k^2) + 24k^2 + 2k^3$$

$$= -4 + 3(h-2k)^2 + 12k^2 + 2k^3$$

$$= -4 + 3(h-2k)^2 + 2k^2(6+k) \quad \text{QED}$$

(B)  $(h-2k)^2 > 0 \quad \forall h, k. \quad 2k^2(6+k) > 0 \quad \text{for } k > -6$

$\therefore$  At P close to A (since small  $h, k \Rightarrow x, y$  close to the  $x, y$ -coords. of A)

The z-word at P is  $p > -4$ ,  $\therefore$  a MINIMUM point at A.

( $\because$  the z-word at A is -4)

(C) B (0, 0, 60) consider planes  $x=0, y=0$ .

$$z = 3x^2 - 12xy + 2y^3 + 60$$

$$z = 2y^3 + 60 \quad \text{when } x=0$$

$z\text{-word} > 60$  for  $y > 0$   
and  $z\text{-word} < 60$  for  $y < 0$

$$z = 3x^2 + 60 \quad \text{when } y=0$$

$z\text{-word} > 60 \quad \forall x$

$\therefore B$  is a SADDLE point.

- (iii) The point Q with coordinates  $(1, 1, 53)$  lies on the surface.

Show that the equation of the tangent plane at Q is

$$6x + 6y + z = 65.$$

[4]

A vector normal to the surface at any point is

$$\underline{n} = \begin{pmatrix} \frac{\partial z}{\partial x} \\ \frac{\partial z}{\partial y} \\ -1 \end{pmatrix} \quad \text{At } (1, 1, 53) \quad \frac{\partial z}{\partial x} = 6x - 12y \\ = 6(1) - 12(1) \\ = -6$$

$$\therefore \begin{pmatrix} -6 \\ -6 \\ -1 \end{pmatrix} \text{ or any multiple of this is normal to the surface at } (1, 1, 53)$$

$$\text{and } \frac{\partial z}{\partial y} = 6y^2 - 12x \\ = 6(1)^2 - 12(1) \\ = -6$$

The EQUATION of the tangent plane is

$$6x + 6y + z = d \Rightarrow 6(1) + 6(1) + 53 = d \\ 65 = d$$

$\therefore$  TANGENT PLANE equation is  $6x + 6y + z = 65$  QED

- (iv) The tangent plane at the point R has equation  $6x + 6y + z = \lambda$  where  $\lambda \neq 65$ .

Find the coordinates of R.

[5]

At R, the normal to the tangent plane is a multiple of  $\begin{pmatrix} 6 \\ 6 \\ 1 \end{pmatrix}$

$$\therefore \begin{pmatrix} 6x - 12y \\ 6y^2 - 12x \\ -1 \end{pmatrix} = \mu \begin{pmatrix} 6 \\ 6 \\ 1 \end{pmatrix} \Rightarrow \mu = -1$$

$$6x - 12y = -6 \quad x - 2y = -1 \quad \textcircled{1} \quad x = 2y - 1$$

$$6y^2 - 12x = -6 \quad y^2 - 2x = -1 \quad \textcircled{2}$$

$$\text{subst. } \textcircled{1} \text{ into } \textcircled{2}: \quad y^2 - 2(2y - 1) + 1 = 0$$

$$y^2 - 4y + 3 = 0$$

$$(y - 3)(y - 1) = 0$$

$$y = 3 \text{ or } y = 1 \quad \text{this is at Q.}$$

$$R \text{ has } x \text{ coord } x = 2(3) - 1 \\ = 5$$

$$\text{and } z \text{ coord } z = 3(5)^2 - 12(5)(3) + 2(3)^3 + 60 \\ = 9$$

$$\therefore R \text{ has coords } (5, 3, 9).$$