

Multivariable Calculus Exam Q

FP3 summer 2013, Q2

A surface has equation $z = 2(x^3 + y^3) + 3(x^2 + y^2) + 12xy$.

- (i) For a point on the surface at which $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y}$, show that either $y = x$ or $y = 1 - x$. [5]

$$\begin{aligned} z &= 2(x^3 + y^3) + 3(x^2 + y^2) + 12xy \\ \frac{\partial z}{\partial x} &= 6x^2 + 6y + 12y \\ \frac{\partial z}{\partial y} &= 6y^2 + 6x + 12x \\ \text{when } \frac{\partial z}{\partial x} &= \frac{\partial z}{\partial y}, \quad 6x^2 + 6y + 12y = 6y^2 + 6x + 12x \\ 6x^2 - 6x + 6y - 6y^2 &= 0 \\ 6x(x-1) - 6y(y-1) &= 0 \quad) \div 6 \\ x^2 - x - y^2 + y &= 0 \\ (x-y)(x+y-1) &= 0 \\ \Rightarrow x=y \quad \text{or} \quad y=1-x & \quad \text{Q.E.D.} \end{aligned}$$

- (ii) Show that there are exactly two stationary points on the surface, and find their coordinates. [7]

$$\begin{aligned} \text{At stat pts, } \frac{\partial z}{\partial x} &= \frac{\partial z}{\partial y} = 0 \\ \frac{\partial z}{\partial x} &= 6x^2 + 6x + 12y \quad \text{subst. } y=1-x \\ \text{subst. } x=y & \quad 0 = 6x^2 + 6x + 12(1-x) \\ 0 &= 6x^2 + 6x + 12 - 12x \\ &= 6x^2 + 18x \\ &= 6x(x+3) \quad 0 = x^2 - x + 2 \quad \text{NO REAL ROOTS.} \\ x=0 \quad \text{or} \quad x=-3 & \quad (\text{DISCRIMINANT } 1-4(2)=-7 < 0) \\ y=0 & \quad y=-3 \\ z=0 & \quad z=2(2(-3)^2) + 3(2(-3)^2) + 12(-3)^2 \\ &= -108 + 54 + 108 \\ &= 54 \end{aligned}$$

$\therefore (0,0,0)$ & $(-3,-3,54)$ are the two stat pts

- (iii) The point $P(\frac{1}{2}, \frac{1}{2}, 5)$ is on the surface, and $Q(\frac{1}{2}+h, \frac{1}{2}+h, 5+w)$ is a point on the surface close to P . Find an approximate expression for h in terms of w . [5]

$$\begin{aligned} \text{At } P, \frac{\partial z}{\partial x} &= 6\left(\frac{1}{2}\right)^2 + 6\left(\frac{1}{2}\right) + 12\left(\frac{1}{2}\right) \\ &= \frac{6}{4} + 3 + 6 \\ &= \frac{21}{2} = \frac{\partial z}{\partial y} \\ \delta z &\approx \frac{\partial z}{\partial x} \delta x + \frac{\partial z}{\partial y} \delta y \\ w &\approx \frac{21}{2} h + \frac{21}{2} h \\ h &\approx \frac{1}{21} w \end{aligned}$$

- (iv) Find the four points on the surface at which the normal line is parallel to the vector $24\mathbf{i} + 24\mathbf{j} - \mathbf{k}$. [7]

The normal line at any point has direction

$$\begin{pmatrix} \frac{\partial z}{\partial x} \\ \frac{\partial z}{\partial y} \\ -1 \end{pmatrix} = \underline{n} \quad \therefore \text{we require } \underline{n} = \lambda \begin{pmatrix} 24 \\ 24 \\ -1 \end{pmatrix}$$

$$\begin{aligned} 6x^2 + 6x + 12y &= 24\lambda \\ 6y^2 + 6y + 12x &= 24\lambda \\ -1 &= -\lambda \end{aligned} \quad \lambda = 1$$

$$\text{from (i)} \quad \frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} \Rightarrow x=y \text{ or } y=1-x$$

$x=y :$

$$\begin{aligned} 6x^2 + 6x + 12x &= 24 \\ 6x^2 + 18x - 24 &= 0 \\ x^2 + 3x - 4 &= 0 \\ (x+4)(x-1) &= 0 \\ x = -4, x = 1 & \quad | \quad y = -x \\ y = -4 & \quad | \quad y = 1 \\ z = 4x^3 + 18x^2 & \quad | \quad z = \end{aligned}$$

using CALC button is
quickest to find z coords

$4x^3 + 18x^2$	$2(x^3 + y^3) + 3(x^2 + y^2) + 12xy$
32	5

\therefore Four pts are: $(-4, -4, 32)$, $(1, 1, 22)$,
 $(-1, 2, 5)$, $(2, -1, 5)$