

2(a)(i) $\cos 5\theta + j \sin 5\theta = (\cos \theta + j \sin \theta)^5$ by de Moivre's thm.
 So $\cos 5\theta$ is the real part of the expansion of $(\cos \theta + j \sin \theta)^5$

Let $\cos \theta = c, \sin \theta = s$

$$(c+js)^5 = c^5 + 5c^4sj + 10c^3(sj)^2 + 10c^2(sj)^3 + 5c(sj)^4 + (sj)^5 \\ = c^5 + 5c^4sj - 10c^3s^2 - 10c^2s^3j + 5cs^4 + s^5j$$

$$\begin{aligned} \operatorname{Re}(c+js)^5 &= c^5 - 10c^3s^2 + 5cs^4 && \text{using } s^2 + c^2 \equiv 1 \\ &= c^5 - 10c^3(1-c^2) + 5c(1-c^2)^2 \\ &= c^5 - 10c^3 + 10c^5 + 5c(1-2c^2+c^4) && \text{show plenty of} \\ &= c^5 - 10c^3 + 10c^5 + 5c - 10c^3 + 5c^5 && \text{working in a} \\ &= 16c^5 - 20c^3 + 5c && \text{"show that" question} \\ &= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta \end{aligned}$$

(ii) $\cos 5\theta = 0 \Rightarrow 16c^5 - 20c^3 + 5c = 0$
 $c(16c^4 - 20c^2 + 5) = 0$

$c \neq 0 \therefore 16c^4 - 20c^2 + 5 = 0$ quadratic in $\cos^2\theta$

using formula with $a=16, b=20, c=5$

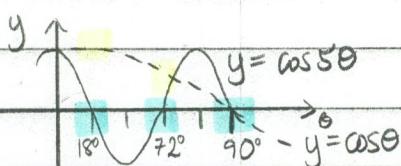
$$c^2 = \frac{20 \pm \sqrt{20^2 - 4(16)(5)}}{2(16)}$$

not asked for a

sketch, but it

really helps sometimes!

$$\cos^2\theta = \frac{5 \pm \sqrt{5}}{8}$$



$$\cos 5\theta = 0 \Rightarrow \theta = 18^\circ, 72^\circ, 90^\circ$$

$$\cos \theta \neq 0 \Rightarrow \theta \neq 90^\circ$$

from sketch, $\cos 18^\circ > \cos 72^\circ$

$$\therefore \cos^2 18^\circ = \frac{5 + \sqrt{5}}{8} \quad \text{and } \cos^2 72^\circ = \frac{5 - \sqrt{5}}{8}$$

$$\Rightarrow \cos 18^\circ = \left(\frac{5}{8} + \frac{\sqrt{5}}{8} \right)^{1/2}$$

✓ take +ve value since $\cos 18^\circ > 0$ you need to justify why this root was chosen

using $\cos^2 18^\circ + \sin^2 18^\circ = 1$

$$S^2 + C^2 \equiv 1$$

$$\begin{aligned} \sin 18^\circ &= (1 - \cos^2 18)^{1/2} \\ &= \left(1 - \frac{5}{8} - \frac{\sqrt{5}}{8} \right)^{1/2} \\ &= \left(\frac{3}{8} - \frac{\sqrt{5}}{8} \right)^{1/2} \end{aligned}$$

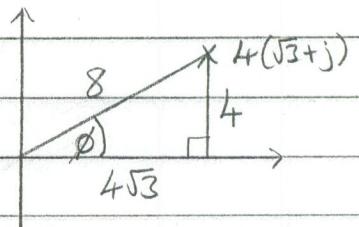
$$(b)(i) 4\sqrt{3} + 4j = 8e^{j\pi/6}$$

cube root modulus,
 $\div \text{arg by } 3$

using "special b/s A"

or $\sqrt{(4\sqrt{3})^2 + 4^2}$

and $\arctan\left(\frac{4}{4\sqrt{3}}\right)$



then add multiples of $\frac{2\pi}{3}$

We were asked for $0 < \theta < 2\pi$

$$r^3 = 8$$

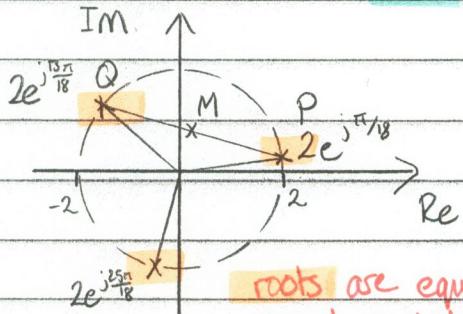
$$r = 2 \quad \theta = 1 \left(\frac{\pi}{3} \right)$$

$$= \frac{\pi}{18}$$

$$\frac{\pi}{18} + \frac{2\pi}{3} \cdot \frac{1}{18}$$

$$\frac{\pi}{18} + \frac{4\pi}{3} \cdot \frac{1}{18}$$

arg MUST be in RADIANS



$$\arg w = \frac{1}{2} (\arg P + \arg Q)$$

\downarrow arg w is average of args of P and Q

$$= \frac{1}{2} \left(\frac{\pi}{18} + \frac{13\pi}{18} \right)$$

$$= \frac{7\pi}{18}$$

roots are equally spread around a circle, centre origin.

For w^n to be real, we need $\arg(w^n)$ to be an integer multiple of π .

$$\arg(w^n) = n \arg w$$

$$\frac{7\pi}{18} n = k\pi$$

$$\underline{n = 18} \quad (k=7)$$

$$\begin{aligned}
 4(i) \quad LHS &= \cosh^2 u - \sinh^2 u \\
 &= \left(\frac{1}{2}(e^u + e^{-u})\right)^2 - \left(\frac{1}{2}(e^u - e^{-u})\right)^2 \\
 &= \frac{1}{4}(e^{2u} + 2e^0 + e^{-2u} - (e^{2u} - 2e^0 + e^{-2u})) \\
 &= \frac{1}{4}(2 - -2) \\
 &= 1 \\
 &= RHS
 \end{aligned}$$

$$(ii) \quad y = \operatorname{arsinh} x$$

$$\sinhy = x$$

$$\cosh y = \frac{dx}{dy}$$

$$\frac{1}{\cosh y} = \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{\pm \sqrt{1+x^2}}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1+x^2}}$$

$$\cosh^2 x - \sinh^2 x \equiv 1$$

$$\Rightarrow \cosh y = \pm \sqrt{1 + \sinh^2 y}$$

$$= \pm \sqrt{1+x^2}$$

but gradient of $\operatorname{arsinh} x$ is always, +ve, \therefore choose only +

show plenty of working in "show that" questions

$$y = \operatorname{arsinh} x$$

$$\sinhy = x$$

$$x = \frac{1}{2}(e^y - e^{-y})$$

$$2x = e^y - e^{-y}$$

$$2e^y x = e^{2y} - 1$$

$$0 = e^{2y} - 2xe^y - 1$$

$$= (e^y - xc)^2 - x^2 - 1$$

$$\pm \sqrt{x^2 + 1} = e^y - xc$$

$$e^y = xc \pm \sqrt{x^2 + 1}$$

$$y = \ln(x \pm \sqrt{x^2 + 1})$$

$$y = \ln(x + \sqrt{x^2 + 1})$$

$$\text{but } xc - \sqrt{x^2 + 1} < 0$$

$$\text{we must have } e^y > 0$$

$$\text{so take +ve root only}$$

$$\begin{aligned}
 \text{(iii)} \quad & \int_0^2 \frac{1}{\sqrt{4+9x^2}} dx = \int_0^2 \frac{1}{\sqrt{2^2+(3x)^2}} dx \quad \text{subst. } u = 3x \\
 & \frac{du}{dx} = 3 \\
 & = \frac{1}{3} \int_0^6 \frac{1}{\sqrt{2^2+u^2}} du = \frac{1}{3} \left[\operatorname{arsinh} \frac{u}{2} \right]_0^6 \\
 & = \frac{1}{3} \operatorname{arsinh} \frac{6}{2} - \frac{1}{3} \operatorname{arsinh} 0 \\
 & = \frac{1}{3} \ln(3 + \sqrt{1+3^2}) - 0 \\
 & = \underline{\frac{1}{3} \ln(3 + \sqrt{10})} \quad \text{must be EXACT form}
 \end{aligned}$$

$$\begin{aligned}
 \text{(iv)} \quad & \int_0^1 \frac{1}{\sqrt{1+x^2}} \operatorname{arsinh} x dx \quad \text{by SUBSTITUTION} \\
 & u = \operatorname{arsinh} x \\
 & \frac{du}{dx} = \frac{1}{\sqrt{1+x^2}} \\
 & du = \frac{1}{\sqrt{1+x^2}} dx \\
 & = \int_0^{\ln(1+\sqrt{2})} u du \\
 & = \left[\frac{1}{2}u^2 \right]_0^{\ln(1+\sqrt{2})} \\
 & = \frac{1}{2}(\ln(1+\sqrt{2}))^2 - 0 \\
 & = \underline{\frac{1}{2}(\ln(1+\sqrt{2}))^2} \quad \text{must be EXACT form}
 \end{aligned}$$

or by PARTS with $u = \operatorname{arsinh} x$

Let the integral be I .

We end up with $I = \left[(\operatorname{arsinh} x)^2 \right]_0^1 - \int_0^1 \frac{1}{\sqrt{1+x^2}} \operatorname{arsinh} x dx$

$$I = \left[(\operatorname{arsinh} x)^2 \right]_0^1 - I$$

$$2I = \left[(\operatorname{arsinh} x)^2 \right]_0^1$$

$$I = \frac{1}{2} \left[(\operatorname{arsinh} x)^2 \right]_0^1$$

$$= \frac{1}{2}(\ln(1+\sqrt{2}))^2$$