

## FP2 January 2008

1. a)  $r = a(1 - \cos 2\theta)$   $0 \leq \theta \leq \pi$ ,  $a > 0$

$$\text{area} = \int_0^\pi \frac{1}{2} a^2 (1 - \cos 2\theta)^2 d\theta$$

$$= \frac{a^2}{2} \int_0^\pi 1 - 2\cos 2\theta + \cos^2 2\theta$$

$$= \frac{a^2}{2} \int_0^\pi 1 - 2\cos 2\theta + \frac{1}{2}\cos 4\theta + \frac{1}{2} d\theta$$

$$= \frac{a^2}{2} \left[ \frac{3}{2}\theta - \sin 2\theta + \frac{1}{8}\sin 4\theta \right]_0^\pi$$

$$= \frac{a^2}{2} \left( \frac{3\pi}{2} - \cancel{\sin 2\pi} + \frac{1}{8}\cancel{\sin 4\pi} - (0 - 0 + 0) \right)$$

$$= \frac{3\pi a^2}{4}$$

let  $2\theta = A$

$$\cos 2A = 2\cos^2 A - 1$$

$$\frac{1}{2}(\cos 2A + 1) = \cos^2 A$$

$$\frac{1}{2}(\cos 4\theta + 1) = \cos^2 2\theta$$

b) i)  $f(x) = \arctan(\sqrt{3} + x)$

$$f'(x) = \frac{1}{1 + (\sqrt{3} + x)^2}$$

$$f''(x) = \frac{0 - 2(\sqrt{3} + x)}{[1 + (\sqrt{3} + x)^2]^2}$$

ii)  $f(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!}$

$$= \arctan \sqrt{3} + \frac{1}{4}x + \frac{-2\sqrt{3}x^2}{16 \cdot 2!} + \dots$$

$$= \frac{\pi}{3} + \frac{x}{4} - \frac{\sqrt{3}x^2}{16} + \dots$$

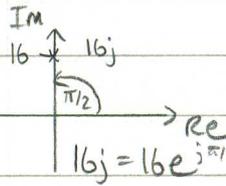


iii)  $\int_{-h}^h x \arctan x dx = \int_{-h}^h \left( \frac{\pi x}{3} + \frac{x^2}{4} - \frac{\sqrt{3}x^3}{16} + \dots \right) dx$

$$= \left[ \frac{\pi x^2}{6} + \frac{x^3}{12} - \frac{\sqrt{3}x^4}{64} \right]_{-h}^h$$
$$= \left( \frac{\pi h^2}{6} + \frac{h^3}{12} - \frac{\sqrt{3}h^4}{64} \right) - \left( \frac{\pi h^2}{6} - \frac{h^3}{12} - \frac{\sqrt{3}h^4}{64} \right) = \frac{2h^3}{12} = \frac{h^3}{6}$$

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2.a)  $z = re^{j\theta}$      $z^4 = r^4 e^{j4\theta} = 16e^{j\pi/2}$



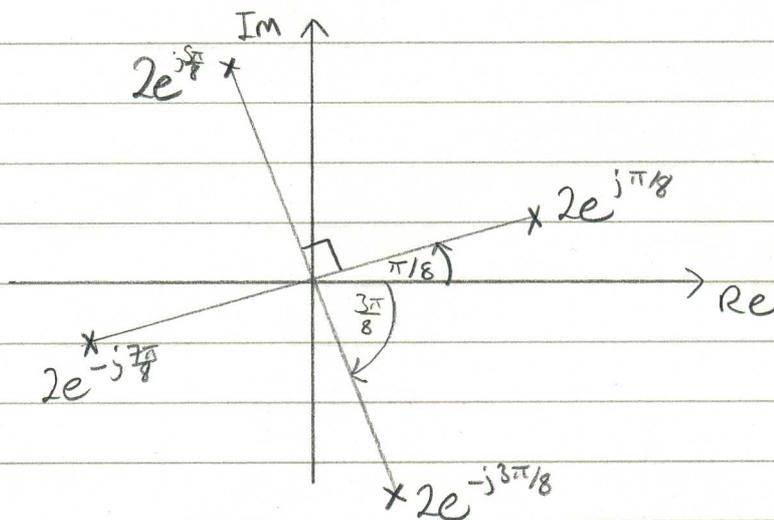
$r^4 = 16$

$4\theta = \pi/2$

$r = 16^{1/4} = 2$

$\theta = \pi/8, \pi/8 + \frac{2\pi}{4}, \pi/8 + \frac{4\pi}{4}, \pi/8 + \frac{6\pi}{4}$   
 $= \frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \frac{13\pi}{8}$

roots are  $2e^{j\pi/8}, 2e^{j5\pi/8}, 2e^{-j7\pi/8}, 2e^{-j3\pi/8}$



b)  $(1 - 2e^{j\theta})(1 - 2e^{-j\theta}) = 1 - 2e^{-j\theta} - 2e^{j\theta} + 4$   
 $= 5 - 2(\cos\theta - j\sin\theta) - 2(\cos\theta + j\sin\theta)$   
 $= 5 - 4\cos\theta$

ii)  $C + jS = 2(\cos\theta + j\sin\theta) + 4(\cos 2\theta + j\sin 2\theta) + 8(\cos 3\theta + j\sin 3\theta) + \dots + 2^n(\cos n\theta + j\sin n\theta)$   
 $= 2e^{j\theta} + 2^2e^{j2\theta} + 2^3e^{j3\theta} + \dots + 2^n e^{jn\theta}$

this is the sum of a GP with  $a = 2e^{j\theta}$ ,  $r = 2e^{j\theta}$

$S_n = \frac{a(1-r^n)}{1-r} = \frac{2e^{j\theta}(1-2^n e^{jn\theta})}{1-2e^{j\theta}} \times \frac{1-2e^{-j\theta}}{1-2e^{-j\theta}}$

$= \frac{2e^{j\theta}(1-2e^{-j\theta}-2^n e^{jn\theta} + 2^{n+1} e^{j(n-1)\theta})}{5-4\cos\theta}$

$= \frac{2e^{j\theta} - 4 - 2^{n+1} e^{j(n+1)\theta} + 2^{n+2} e^{j(n+2)\theta}}{5-4\cos\theta}$

equating Re:  $C = (2\cos\theta - 4 - 2^{n+1}\cos((n+1)\theta) + 2^{n+2}\cos(n\theta)) / (5-4\cos\theta)$

equating Im:  $S = (2\sin\theta - 2^{n+1}\sin((n+1)\theta) + 2^{n+2}\sin(n\theta)) / (5-4\cos\theta)$

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3. i)  $M = \begin{pmatrix} 7 & 3 \\ -4 & -1 \end{pmatrix}$  char. eqn  $|M - \lambda I| = 0$

$$\begin{vmatrix} 7-\lambda & 3 \\ -4 & -1-\lambda \end{vmatrix} = (7-\lambda)(-1-\lambda) + 12 = 0$$

$$0 = -7 - 7\lambda + \lambda + \lambda^2 + 12 = 0$$

$$= \lambda^2 - 6\lambda + 5$$

$$= (\lambda - 5)(\lambda - 1) \Rightarrow \text{eigenvalues } \lambda_1 = 5, \lambda_2 = 1$$

$$(M - \lambda_1 I) \underline{s}_1 = 0$$

$$2x + 3y = 0 \quad \textcircled{1} \Rightarrow \text{same eqns}$$

$$-4x - 6y = 0 \quad \textcircled{2} \quad \therefore y = -2/3x$$

$$\underline{s}_1 = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \quad \lambda_1 = 5$$

$$(M - \lambda_2 I) \underline{s}_2 = 0$$

$$6x + 3y = 0 \quad \textcircled{1} \text{ same eqn}$$

$$-4x - 2y = 0 \quad \textcircled{2} \quad \therefore y = -2x$$

$$\underline{s}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad \lambda_2 = 1$$

ii)  $P^{-1}MP = D \quad M = PDP^{-1} \quad P = (\underline{s}_1 \ \underline{s}_2) \quad D = \Lambda$

$$P = \begin{pmatrix} 3 & 1 \\ -2 & -2 \end{pmatrix} \quad = \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix}$$

$$P^{-1} = \frac{-1}{4} \begin{pmatrix} -2 & -1 \\ 2 & 3 \end{pmatrix}$$

$$M^n = P D^n P^{-1} = \begin{pmatrix} 3 & 1 \\ -2 & -2 \end{pmatrix} \begin{pmatrix} 5^n & 0 \\ 0 & 1 \end{pmatrix} \times \frac{-1}{4} \begin{pmatrix} -2 & -1 \\ 2 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 3 \times 5^n & 1 \\ -2 \times 5^n & -2 \end{pmatrix} \times \frac{-1}{4} \begin{pmatrix} -2 & -1 \\ 2 & 3 \end{pmatrix}$$

$$= \frac{-1}{4} \begin{pmatrix} -6 \times 5^n + 2 & -3 \times 5^n + 3 \\ 4 \times 5^n - 4 & 2 \times 5^n - 6 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{3}{2} \times 5^n - \frac{1}{2} & -\frac{3}{4} \times 5^n + \frac{3}{4} \\ -5^n + 1 & -\frac{1}{2} \times 5^n + \frac{3}{2} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\therefore \underline{a = \frac{3}{2} \times 5^n - \frac{1}{2}}$$

$$\underline{b = -\frac{3}{4} \times 5^n + \frac{3}{4}}$$

$$\underline{c = -5^n + 1}$$

$$\underline{d = -\frac{1}{2} \times 5^n + \frac{3}{2}}$$

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4. i)  $k \geq 1$ ,  $\cosh x = k$        $x = \operatorname{arcosh} k$

$$\frac{1}{2}(e^x + e^{-x}) = k \quad \left| \times 2e^x \right.$$

$$e^{2x} + 1 = 2ke^x$$

$$(e^x)^2 - 2ke^x + 1 = 0 \quad \text{quadratic in } e^x, \quad a=1, b=-2k, c=1$$

$$e^x = \frac{2k \pm \sqrt{4k^2 - 4}}{2} \quad \text{by formula}$$

$$e^x = k \pm \sqrt{k^2 - 1}$$

$$x = \ln(k \pm \sqrt{k^2 - 1})$$

We have  $x = \ln(k + \sqrt{k^2 - 1})$ , need to show that

$$x = \ln(k - \sqrt{k^2 - 1}) = -\ln(k + \sqrt{k^2 - 1}) = \ln\left(\frac{1}{k + \sqrt{k^2 - 1}}\right)$$

we need  $k - \sqrt{k^2 - 1} = \frac{1}{k + \sqrt{k^2 - 1}}$

$$\begin{aligned} (k + \sqrt{k^2 - 1})(k - \sqrt{k^2 - 1}) &= k^2 - \sqrt{k^2 - 1}^2 \\ &= k^2 - (k^2 - 1) \\ &= 1 \quad \text{as req'd.} \end{aligned}$$

$\therefore x = \pm \ln(k + \sqrt{k^2 - 1})$

ii)  $\int_1^2 \frac{1}{\sqrt{4x^2 - 1}} dx = \int_1^2 \frac{1}{\sqrt{4(x^2 - (\frac{1}{2})^2)}} dx = \frac{1}{2} \int_1^2 \frac{1}{\sqrt{x^2 - (\frac{1}{2})^2}} dx$

$$= \frac{1}{2} [\operatorname{arcosh}(2x)]_1^2 = \frac{1}{2} (\operatorname{arcosh} 4 - \operatorname{arcosh} 2)$$

$$= \frac{1}{2} \ln(4 + \sqrt{4^2 - 1}) - \frac{1}{2} \ln(2 + \sqrt{2^2 - 1})$$

$$= \frac{1}{2} \ln\left(\frac{4 + \sqrt{15}}{2 + \sqrt{3}}\right)$$

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4. iii)  $6 \sinh x - \sinh 2x = 0$

use identity

$$6 \sinh x - 2 \sinh x \cosh x = 0$$

$$\sinh 2x = 2 \sinh x \cosh x$$

$$2 \sinh x (3 - \cosh x) = 0$$

$$\sinh x = 0 \quad \text{or} \quad \cosh x = 3$$

$$\underline{x = 0} \quad \text{or} \quad x = \pm \ln(3 + \sqrt{3^2 - 1})$$
$$= \pm \ln(3 + \sqrt{8})$$

grad. is  $\frac{dy}{dx} = 6 \cosh x - 2 \cosh 2x$

if grad. = 5 we would have

$$6 \cosh x - 2 \cosh 2x = 5$$

use identity

$$6 \cosh x - 2(2 \cosh^2 x - 1) - 5 = 0$$

$$\cosh 2x = 2 \cosh^2 x - 1$$

$$6 \cosh x - 4 \cosh^2 x - 3 = 0$$

quadratic in  $\cosh x$  with  $a = 4$ ,  $b = -6$ ,  $c = 3$

$$\text{discriminant } b^2 - 4ac = 36 - 4(4)(3) = -12 < 0$$

$\therefore$  no solutions to equation

$\therefore$  grad. is never equal to 5.