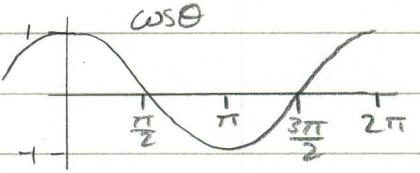


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I. a) $r = a(1 - \cos\theta)$, $a > 0$

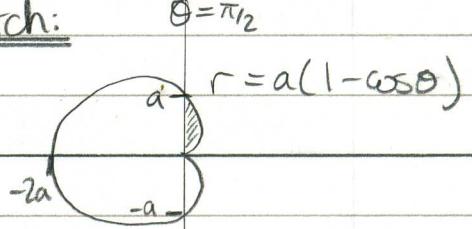


θ	r
$0 \rightarrow \frac{\pi}{2}$	$0 \rightarrow a$
$\frac{\pi}{2} \rightarrow \pi$	$a \rightarrow -a$

$\cos\theta$ even function

so use symmetry to complete $\theta = \pi$ in other two quadrants

sketch:



ii) Area = $\int_0^{\pi/2} \frac{1}{2} a^2 (1 - \cos\theta)^2 d\theta$

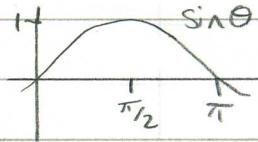
$$\cos 2\theta = 2\cos\theta - 1$$

$$\frac{1}{2}(\cos 2\theta + 1) = \cos^2\theta$$

$$= \frac{a^2}{2} \int_0^{\pi/2} 1 - 2\cos\theta + \cos^2\theta d\theta$$

$$= \frac{a^2}{2} \int_0^{\pi/2} 1 - 2\cos\theta + \frac{1}{2}\cos 2\theta + \frac{1}{2} d\theta$$

$$= \frac{a^2}{2} \left[\frac{3\theta}{2} - 2\sin\theta + \frac{1}{4}\sin 2\theta \right]_0^{\pi/2}$$



$$= \frac{a^2}{2} \left(\left(\frac{3\pi}{4} - 2\sin\frac{\pi}{2} + \frac{1}{4}\sin\pi \right) - (0 - 0 + 0) \right)$$

$$= \frac{a^2}{2} \left(\frac{3\pi}{4} - 2 \right) \text{ square units}$$

b) $\int_0^1 \frac{1}{(4 - x^2)^{3/2}} dx = \int_0^1 \frac{1}{(4(1 - (\frac{x}{2})^2)^{3/2}} dx$

$$\text{subst. } \frac{x}{2} = \sin\theta$$

$$= \frac{1}{4^{3/2}} \int_0^{\pi/6} \frac{1}{(1 - \sin^2\theta)^{3/2}} \times 2\cos\theta d\theta$$

$$x = 2\sin\theta$$

$$= \frac{1}{4} \int_0^{\pi/6} \frac{\cos\theta}{(\cos^2\theta)^{3/2}} d\theta = \frac{1}{4} \int_0^{\pi/6} \sec^2\theta d\theta$$

$$\frac{dx}{d\theta} = 2\cos\theta$$

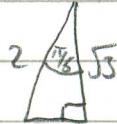
$$= \frac{1}{4} [\tan\theta]_0^{\pi/6} = \frac{1}{4} (\tan\pi/6 - \tan 0) = \frac{1}{4} \times \frac{1}{\sqrt{3}}$$

$$\text{limits: } x=0, \theta=0$$

$$= \frac{1}{4\sqrt{3}}$$

$$x=1, \sin\theta=\frac{1}{2}$$

$$\theta=\frac{\pi}{6}$$



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$$1. \text{ c) } f(x) = \arccos(2x) \quad i) \quad f'(x) = \frac{-2}{\sqrt{1-(2x)^2}} = \frac{-2}{\sqrt{1-4x^2}}$$

$$\text{ii) } f'(x) = -2(1-4x^2)^{-1/2}$$

$$= -2 \left(1 + \frac{(-1/2)(-4x^2)}{1!} + \frac{(-1/2)(-3/2)(-4x^2)^2}{2!} + \dots \right)$$

$$= -2(1 + 2x^2 + 6x^4 + \dots)$$

$$= -2 - 4x^2 - 12x^4$$

$$f(x) = \int f'(x) dx = -2x - \frac{4x^3}{3} - \frac{12x^5}{5} + \dots + C$$

$$f(0) = \frac{\pi}{2} = C$$

$$f(x) = \frac{\pi}{2} - 2x - \frac{4x^3}{3} - \frac{12x^5}{5} + \dots$$

$$2. \quad (\cos\theta + j\sin\theta)^5 = \cos 5\theta + j\sin 5\theta$$

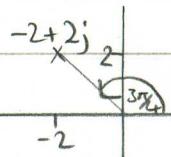
$$\text{let } \cos\theta = c, \sin\theta = s$$

$$\begin{aligned} \text{bin. exp.} \quad & c^5 + j5c^4s + 10j^2c^3s^2 + 10j^3c^2s^3 + 5j^4cs^4 + j^5s^5 \\ & = c^5 + j5c^4s - 10c^3s^2 - j10c^2s^3 + 5cs^4 + j5s^5 \end{aligned}$$

$$\begin{aligned} \text{eq. Im: } \sin 5\theta &= 5(1-s^2)^2s - 10(1-s^2)s^3 + s^5 \\ &= 5(1-2s^2+s^4)s - 10s^3 + 10s^5 + s^5 \\ &= 5s - 10s^3 + 5s^5 - 10s^3 + 11s^5 \\ &= 5\sin\theta - 20\sin^3\theta + 16\sin^5\theta \end{aligned}$$

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2. b) i) $z = r e^{j\theta}$ $z^3 = r^3 e^{j3\theta} = -2 + 2j$
 $= \sqrt{8} e^{j3\pi/4}$



$$r^3 = \sqrt{8}$$

$$3\theta = \frac{3\pi}{4}$$

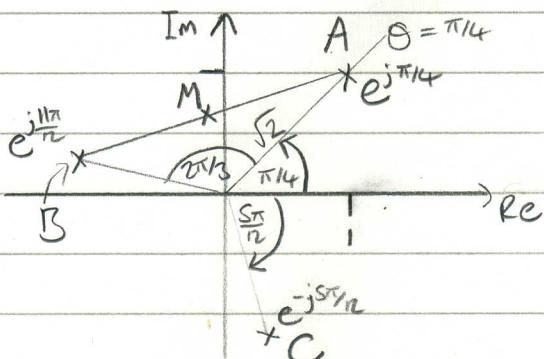
$$|-2+2j| = \sqrt{2^2 + 2^2} = \sqrt{8}$$

$$\theta = \frac{\pi}{4} \quad \text{or} \quad \frac{\pi}{4} + \frac{2\pi}{3} = \frac{11}{12}\pi$$

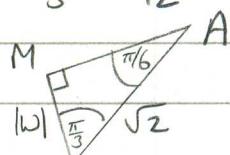
$$\text{or } \frac{\pi}{4} - \frac{2\pi}{3} = -\frac{5}{12}\pi$$

roots are $\underline{\sqrt{2}e^{-j\frac{5\pi}{12}}}, \underline{\sqrt{2}e^{j\frac{\pi}{12}}}, \underline{\sqrt{2}e^{j\frac{11\pi}{12}}}$

ii)



$$M: \arg(w) = \frac{\pi}{4} + \frac{\pi}{3} = \frac{7\pi}{12}$$



$$|w| = \sqrt{2} \sin \frac{\pi}{6}$$

iii) $\underline{\arg w = \frac{7\pi}{12}}, \underline{|w| = \sqrt{2}/2}$

iv) $w = \frac{\sqrt{2}}{2} e^{j\frac{7\pi}{12}} \Rightarrow w^6 = \left(\frac{\sqrt{2}}{2}\right)^6 e^{j42\pi/12}$

$$= \frac{8}{64} e^{j\frac{7\pi}{2}} = \frac{1}{8} e^{-j\pi/2}$$

$$= \frac{1}{8} (\cos(-\pi/2) + j \sin(-\pi/2))$$

$$= \underline{-\frac{1}{8}j}$$

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3. i) $M - \lambda I = \begin{pmatrix} 3-\lambda & 5 & 2 \\ 5 & 3-\lambda & -2 \\ 2 & -2 & -4-\lambda \end{pmatrix}$ char. eqn. $|M - \lambda I| = 0$

$$\begin{pmatrix} + & - & + \\ + & - & + \end{pmatrix}$$

$$\begin{aligned} |M - \lambda I| &= (3-\lambda)((3-\lambda)(4-\lambda) - 4) - 5(5(-4-\lambda) + 4) + 2(-10 - 2(3-\lambda)) \\ &= (3-\lambda)(-12 - 3\lambda + 4\lambda + \lambda^2 - 4) - 5(-20 - 5\lambda + 4) + 2(-10 - 6 + 2\lambda) \\ &= (3-\lambda)(-16 + \lambda + \lambda^2) + 80 + 25\lambda - 32 + 4\lambda \\ &= -48 + 3\lambda + 3\lambda^2 + 16\lambda - \lambda^2 - \lambda^3 + 48 + 29\lambda \\ 0 &= 48\lambda + 2\lambda^2 - \lambda^3 \\ 0 &= \lambda^3 - 2\lambda^2 - 48\lambda \end{aligned}$$

ii) $\lambda_1 = 0, \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ $0 = \lambda(\lambda^2 - 2\lambda - 48)$ $\lambda_2 = 8, \lambda_3 = -6$

$$= \lambda(\lambda - 8)(\lambda + 6)$$

$$(M - \lambda_1 I) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$$-5x + 5y + 2z = 0 \quad \textcircled{1} \quad \left. \begin{array}{l} \text{same eqn.} \\ \textcircled{2} \end{array} \right\}$$

$$5x - 5y - 2z = 0 \quad \textcircled{2} \quad \textcircled{2} \text{ and } \textcircled{3} \text{ inconsistent}$$

$$2x - 2y - 12z = 0 \quad \textcircled{3} \quad \therefore z = 0, x = y$$

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$(M - \lambda_2 I) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$$9x + 5y + 2z = 0 \quad \textcircled{1} \quad \textcircled{1} + \textcircled{2}: 14x + 14y = 0$$

$$5x + 9y - 2z = 0 \quad \textcircled{2} \quad \textcircled{2} + \textcircled{3}: 7x + 7y = 0$$

$$2x - 2y + 2z = 0 \quad \textcircled{3} \quad \therefore x = -y$$

$$\text{subst. into } \textcircled{1}: 9x - 5x + 2z = 0$$

$$4x + 2z = 0 \quad 2x = -z$$

$$\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$$

eigenvalues 0, 8, -6 corresponding to eigenvectors $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$

iii) $P^{-1} M^2 P = D \quad M^2 = P D P^{-1} \quad \text{where } P = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & -1 \\ 1 & 0 & -2 \end{pmatrix}, D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 64 & 0 \\ 0 & 0 & 36 \end{pmatrix}$

$$M^2 = S \Lambda^2 S^{-1}$$

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i) by C-H thm, $M^3 - 2M^2 - 48M = 0$

$$M^3 = 2M^2 + 48M$$

$$M^4 = MM^3 = M(2M^2 + 48M)$$

$$= 2M^3 + 48M^2$$

$$= 2(2M^2 + 48M) + 48M^2$$

$$= \underline{\underline{52M^2 + 96M}}$$

$a = 52, b = 96$

4. a) $\int_0^1 \frac{1}{\sqrt{9x^2 + 16}} dx = \int_0^1 \frac{1}{\sqrt{(3x)^2 + 4^2}} dx$

$$= \frac{1}{3} [\operatorname{arsinh} \frac{3x}{4}]_0^1 = \frac{1}{3} \operatorname{arsinh} \frac{3}{4} - \frac{1}{3} \operatorname{arsinh} 0$$

$$= \frac{1}{3} \ln \left(\frac{3}{4} + \sqrt{\frac{9}{16} + 1} \right) = \frac{1}{3} \ln \left(\frac{3}{4} + \frac{5}{4} \right) = \underline{\underline{\frac{1}{3} \ln 2}}$$

b) i) $2 \sinh x \cosh x = 2 \times \frac{1}{2} (e^x - e^{-x})(e^x + e^{-x})$

$$= \frac{1}{2} (e^{2x} - e^{-2x})$$

$$= \sinh 2x$$

ii) $\frac{dy}{dx} = 0$ at stat. pts $y = 20 \cosh x - 3 \cosh 2x$

$$\frac{dy}{dx} = 20 \sinh x - 6 \sinh 2x = 0$$

$$20 \sinh x - 6 \times 2 \sinh x \cosh x = 0$$

$$4 \sinh x (5 - 3 \cosh x) = 0$$

$$\Rightarrow \sinh x = 0 \quad \text{or} \quad \cosh x = \frac{5}{3}$$

$$x = 0$$

$$x = \operatorname{arccosh} \frac{5}{3} = \pm \ln \left(\frac{5}{3} + \sqrt{\frac{25}{9} - 1} \right)$$

$$= \pm \ln \left(\frac{5}{3} + \frac{4}{3} \right) = \ln 3 \text{ or } -\ln 3$$

$$y = 20 \cosh 0 - 3 \cosh 0$$

$$= 20 - 3$$

$$= 17$$

$$y = 20 \cosh(\ln 3) - 3 \cosh(2 \ln 3)$$

$$= \frac{59}{3} \quad \text{also } y = \frac{59}{3} \text{ for } -\ln 3$$

coords: (0, 17), (\ln 3, \frac{59}{3}), (-\ln 3, \frac{59}{3})

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$$\begin{aligned}
 4.b) iii) & \int_{-\ln 3}^{\ln 3} 20 \cosh x - 3 \cosh 2x \, dx \\
 &= [20 \sinh x - \frac{3}{2} \sin 2x]_{-\ln 3}^{\ln 3} \\
 &= 20 \sinh(\ln 3) - \frac{3}{2} \sin(\ln 9) - (20 \sinh(-\ln 3) - \frac{3}{2} \sinh(-\ln 3)) \\
 &= \left[\frac{20}{2} (3 - \frac{1}{3}) - \frac{3}{4} (9 - \frac{1}{9}) \right] \times 2 \\
 &= \left(\frac{80}{3} - \frac{20}{3} \right) \times 2 \\
 &= \underline{\underline{40}}
 \end{aligned}$$

for part ii) on previous page, show the fractions from your working:

$$\begin{aligned}
 y &= 20 \cosh(\ln 3) - 3 \cosh(2 \ln 3) \\
 &= 20 \times \frac{1}{2} (e^{\ln 3} + e^{-\ln 3}) - 3 \times \frac{1}{2} (e^{\ln 9} + e^{-\ln 9}) \\
 &= 10(3 + \frac{1}{3}) - \frac{3}{2}(9 + \frac{1}{9}) \\
 &= \frac{100}{3} - \frac{1}{2}(27 + \frac{1}{3}) \\
 &= \frac{100}{3} - \frac{41}{3} \\
 &= \underline{\underline{\frac{59}{3}}}
 \end{aligned}$$