FP1 Summer 2015

1 $\underline{M} = \begin{pmatrix} 4 & -3 \\ 8 & 21 \end{pmatrix}$ $\quad \underline{M}^{-1} = \frac{1}{\Delta}\begin{pmatrix} 21 & 3 \\ -8 & 4 \end{pmatrix}$ $\quad \Delta = 4(21) - 8(-3)$

$= \frac{1}{108}\begin{pmatrix} 21 & 3 \\ -8 & 4 \end{pmatrix}$ $\quad = 84 + 24$

$= 108$

$\underline{M}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$\underline{M}^{-1}\,\underline{M}\begin{pmatrix} x \\ y \end{pmatrix} = \underline{M}^{-1}\begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{108}\begin{pmatrix} 21 & 3 \\ -8 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$= \begin{pmatrix} \frac{5}{18} \\ \frac{1}{27} \end{pmatrix}$

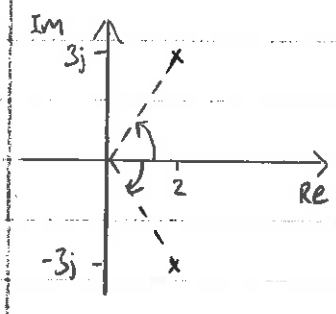
$\therefore \underline{\underline{x = \frac{5}{18}, \; y = \frac{1}{27}}}$

2 $z^2 - 4z + 13 = 0$

$(z-2)^2 - 4 = -13$

$z - 2 = \pm\sqrt{-9}$

$\underline{\underline{z = 2 \pm 3j}}$



$|2+3j| = \sqrt{2^2 + 3^2}$

$= \underline{\underline{\sqrt{13}}}$

$|2-3j| = \underline{\underline{\sqrt{13}}}$

$\arg(2+3j) = \arctan\left(\frac{3}{2}\right)$

$= \underline{\underline{0.983 \text{ rad}}}$ (3s.f.)

$\arg(2-3j) = \underline{\underline{-0.983^c}}$ (3s.f.)

3 $2x^3 + px^2 + qx + r = 0$ $\quad a=2, \; b=p, \; c=q, \; d=r$

$\Sigma\alpha = -\frac{p}{2} = 6$ $\quad \alpha\beta\gamma = -\frac{r}{2} = -10$

$\underline{\underline{p = -12}}$ $\quad \underline{\underline{r = 20}}$

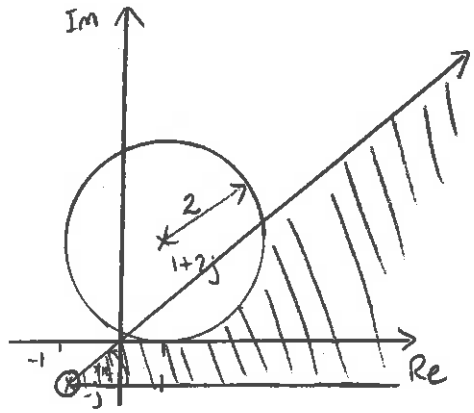
$x = 4$ is a root $\Rightarrow 2(4)^3 - 12(4)^2 + 4q + 20 = 0$

$128 - 192 + 4q + 20 = 0$

$4q = 44$

$\underline{\underline{q = 11}}$

4



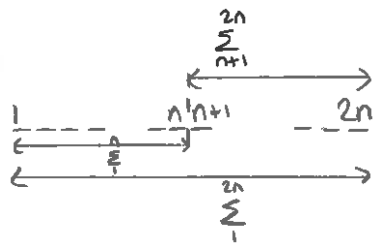
(i) the halfline from point $-1-j$ in the $\frac{\pi}{4}$ direction (goes thro' ORIGIN)

(ii) circle, centre $1+2j$, radius 2

(iii) shaded region

5 (i)
$$\sum_{r=1}^{n} 2r-1 = 2\sum_{r=1}^{n} r - \sum_{r=1}^{n} 1$$
$$= \cancel{2}\left(\tfrac{1}{\cancel{2}}\right)n(n+1) - n$$
$$= n^2 + n - n$$
$$= \underline{\underline{n^2}} \quad \text{Q.E.D.}$$

(ii)
$$\sum_{r=n+1}^{2n} 2r-1 = \sum_{r=1}^{2n} 2r-1 - \sum_{r=1}^{n} 2r-1$$
$$= (2n)^2 - n^2$$
$$= 3n^2$$



$$\therefore \frac{\sum_{r=1}^{n} 2r-1}{\sum_{r=n+1}^{2n} 2r-1} = \frac{n^{\cancel{2}}}{3\cancel{n^2}}$$
$$= \underline{\underline{\frac{1}{3}}} = k$$

6 $u_1 = 3 \quad u_{n+1} = 3u_n - 5 \qquad u_n = \dfrac{3^{n-1}+5}{2}$

let $n=1$, inductive formula gives $u_1 = 3$

"n^{th} term formula" gives
$$u_1 = \frac{3^{1-1}+5}{2}$$
$$= \frac{6}{2}$$
$$= 3$$

both give same value, ∴ true for $n=1$.

6 cont'd

Assume true for $n=k$, so $u_k = \dfrac{3^{k-1}+5}{2}$

subst. into inductive formula for u_{k+1}:

$$u_{k+1} = 3\left(\frac{3^{k-1}+5}{2}\right) - 5$$

$$= \frac{3^1 \times 3^{k-1} + 3(5) - 5(2)}{2}$$

$$= \frac{3^k + 15 - 10}{2}$$

$$= \frac{3^k + 5}{2}$$

n^{th} term formula for u_{k+1} is:

$$u_{k+1} = \frac{3^{k+1-1}+5}{2}$$

$$= \frac{3^k + 5}{2}$$

both give same expression.

$\therefore$ If it is true for $n=k$, then it is true for $n=k+1$.
As it is true for $n=1$, it is true for all $n \geq 1$.

7 (i) $y = \dfrac{(3x+2)(x-3)}{(x-2)(x+1)}$

vert. asym. where denom. $=0$

$\therefore$ $\underline{\underline{x=2}}$, $\underline{\underline{x=-1}}$

horiz. asym. as $x \to \infty$, $y \to \dfrac{3\cancel{x^2}}{\cancel{x^2}} = 3$ $\therefore$ $\underline{\underline{y=3}}$

y-int. where $x=0$: $y = \dfrac{2(-3)}{-2(1)}$

$= 3$ $\therefore$ $\underline{\underline{(0,3)}}$

x-int where $y=0$: $0 = (3x+2)(x-3)$

$x = -\frac{2}{3}$, $x = 3$

$\therefore$ $\underline{\underline{(-\frac{2}{3}, 0)}}$, $\underline{\underline{(3,0)}}$

(ii) subst $x = 100$, $y = \dfrac{302(97)}{98(101)} = 2.959\ldots < 3$

$\therefore$ as $\underline{\underline{x \to \infty,\ y \to 3 \text{ from below}}}$

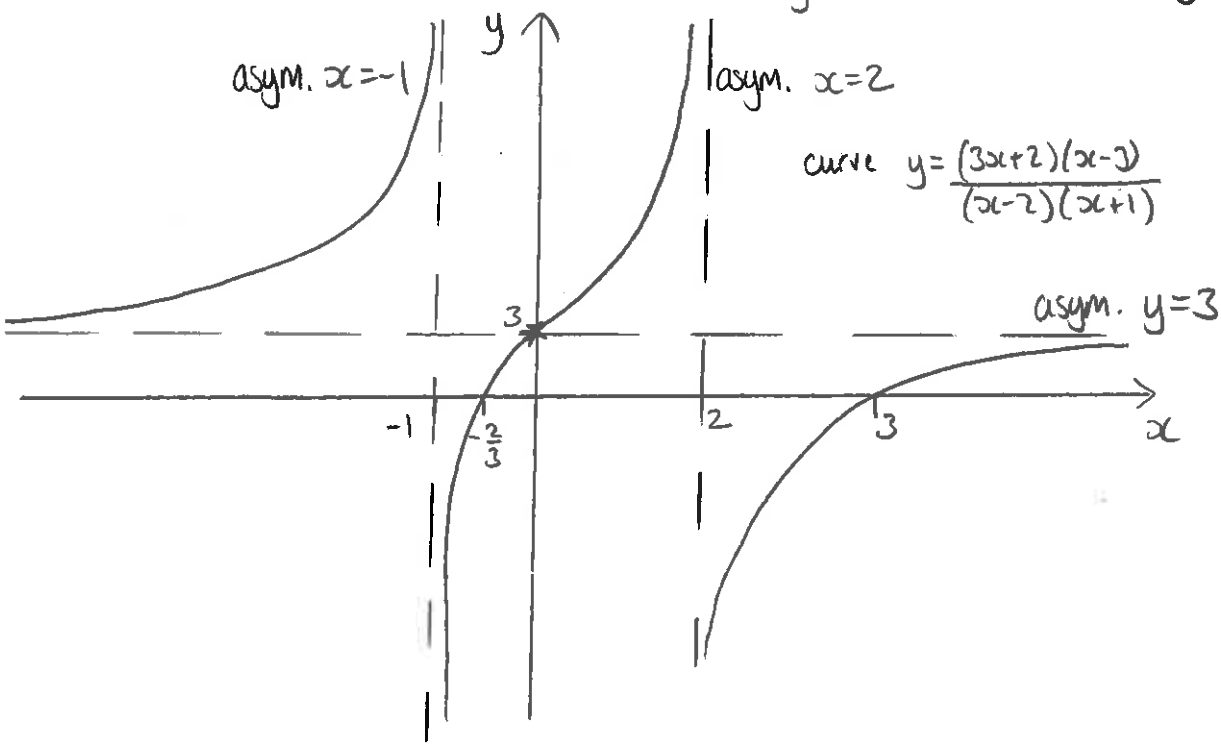
7 cont'd subst. $x = -100$: $y = \dfrac{-298(-103)}{-102(-99)} = 3.039\ldots > 3$

$\therefore$ as $x \to -\infty$, $y \to 3$ from above.

behaviour near vert. asym

$x = -1$: LHS $x = -1.1$, $y > 0$; RHS $x = -0.9$, $y < 0$

$x = 2$: LHS $x = 1.9$, $y > 0$; RHS $x = 2.1$, $y < 0$



(iii) from sketch, $y \geq 3$ for $x < -1$, $0 \leq x < 2$

8 (i) $\alpha = 5 + 4j$

$$\alpha^2 = (5+4j)(5+4j)$$
$$= 25 + 20j + 20j + 16j^2$$
$$= 25 + 40j - 16$$
$$= 9 + 40j$$

$j^2 = -1$

$$\alpha^3 = (9+40j)(5+4j)$$
$$= 45 + 36j + 200j + 160j^2$$
$$= 45 + 236j - 160$$
$$= -115 + 236j$$

$j^2 = -1$

8 cont'd

(ii) $\alpha^3 + q\alpha^2 + 11\alpha + r = 0$

$-115 + 236j + q(9+40j) + 11(5+4j) + r = 0$

equate Re parts: $-115 + 9q + 55 + r = 0$

$9q + r = 60$ ①

equate Im. parts: $236 + 40q + 44 = 0$

$40q = -280$

$\underline{\underline{q = -7}}$

subst. q in ①:

$9(-7) + r = 60$

$\underline{\underline{r = 123}}$

$f(z) = z^3 - 7z^2 + 11z + 123$

$\underline{\underline{5+4j}}$ is a root of $f(z)=0$ $\therefore$ its conjugate $\underline{\underline{5-4j}}$ is also a root

$\therefore$ $(z-(5+4j))$ and $(z-(5-4j))$ are factors of $f(z)$

$f(z) = (z-5-4j)(z-5+4j)(z-a)$

constant term is $(25-16j^2)(-a) = -41a = 123$

$\Rightarrow \underline{\underline{z = -3}}$

ALTERNATIVELY, using $\Sigma\alpha = 7$,

$5\cancel{+4j} + 5\cancel{-4j} + a = 7$

$a = -3$

the 3 roots are

$\underline{\underline{z = 5 \pm 4j,\ z = -3}}$

(iv) $z^4 + qz^3 + 11z^2 + rz = z^3 + qz^2 + 11z + r$

$zf(z) = f(z)$

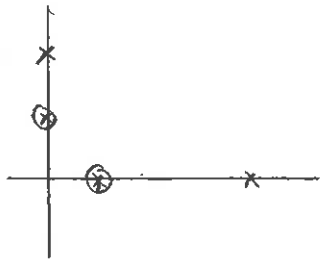
$\underline{\underline{z = 1,\ z = -3,\ z = 5+4j,\ z = 5-4j}}$

9 (i)

$$\begin{array}{cc} & \begin{array}{ccc} A & B & C \end{array} \\ \begin{pmatrix} 1 & -2 \\ 3 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 4 \\ 0 & 2 & 1 \end{pmatrix} \end{array} = \begin{array}{c} \begin{array}{ccc} A' & B' & C' \end{array} \\ \begin{pmatrix} 0 & -4 & -2 \\ 0 & 0 & 12 \end{pmatrix} \end{array}$$

$\therefore$ coords $\underline{\underline{A'(0,0)}}$; $\underline{\underline{B'(-4,0)}}$; $\underline{\underline{C'(-2,12)}}$

(ii)



$\underline{M}$ represents a two-way STRETCH, of scale factor 4 // to x-axis and scale factor 2 // to y-axis.

$\underline{M}\,\underline{T}$ maps $\triangle ABC$ onto $\triangle A''B''C''$. We require inverse matrix $(\underline{M}\,\underline{T})^{-1}$

$$\underline{M}\,\underline{T} = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 3 & 0 \end{pmatrix} = \begin{pmatrix} 4 & -8 \\ 6 & 0 \end{pmatrix}$$

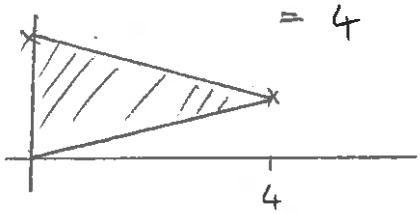
$$\det(\underline{M}\,\underline{T}) = 0 - -48 = 48$$

$$(\underline{M}\,\underline{T})^{-1} = \underline{\underline{\frac{1}{48}\begin{pmatrix} 0 & 8 \\ -6 & 4 \end{pmatrix}}}$$

this is the single transformation matrix required.

(iii) $\det(\underline{M}\,\underline{T})$ is the area scale factor of the transformation.

$\triangle ABC$ area $= \frac{1}{2}(4)(2) = 4$



$\therefore$ area of $\triangle A''B''C''$ is $4 \times 48 = \underline{\underline{192 \text{ sq. units}}}$