

C4 2016 Paper A

$$1 \quad \cos\theta - 3\sin\theta = R\cos(\theta + \alpha) \\ = R\cos\theta\cos\alpha - R\sin\theta\sin\alpha$$

equate coeffs $\cos\theta$: $1 = R\cos\alpha \quad R = \sqrt{1^2 + 3^2}$
 " " " $\sin\theta$: $3 = R\sin\alpha \quad = \sqrt{10}$

$$\frac{R\sin\alpha}{R\cos\alpha} = \frac{3}{1} \quad \alpha = \arctan(3) = 1.249 \text{ (4 s.f.)}$$

$$\therefore \cos\theta - 3\sin\theta \equiv \sqrt{10} \cos(\theta + 1.249)$$

This has max. value $\sqrt{10}$ and $\sqrt{10} < 4$, \therefore
 $\cos\theta - 3\sin\theta = 4$ has no solution.

$$2 \quad \left(1 + \frac{x}{p}\right)^q = 1 + qx + \frac{3}{4}x^2 + \dots$$

$$\left(1 + \frac{x}{p}\right)^q = 1 + q\left(\frac{x}{p}\right) + \frac{q(q-1)\left(\frac{x}{p}\right)^2}{2!} + \dots$$

equate coeffs of x : $-1 = \frac{q}{p} \quad ①$

" " " x^2 : $\frac{3}{4} = \frac{q(q-1)}{2p^2} \quad ②$

from ①, $p = -q$, subst. in ②: $\frac{3}{4} = \frac{q(q-1)}{2(-q)^2}$

$$6q^2 = 4q^2 - 4q$$

$$2q^2 + 4q = 0$$

$$q(q+2) = 0$$

$$q \neq 0 \therefore q = -2, p = 2$$

valid for $\left|\frac{x}{2}\right| < 1 \Rightarrow |x| < 2$

$$3 \quad y = x^4 \quad \text{VOLUME} = \pi \int_0^4 x^2 \, dy$$

$$y^{1/2} = x^2 \quad = \pi \int_0^4 y^{1/2} \, dy$$

$$= \pi \left[\frac{2}{3} y^{3/2} \right]_0^4$$

$$= \pi \left(\frac{2}{3} (4^{3/2}) - 0 \right)$$

$$= \underline{\underline{\frac{16\pi}{3}}} \text{ cubic units}$$

$$4 \quad 2 \sin 2\theta = 1 + \cos 2\theta$$

DOUBLE & FORMULAE

$$2(2\sin\theta\cos\theta) = 1 + 2\cos^2\theta - 1$$

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$4\sin\theta\cos\theta - 2\cos^2\theta = 0$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$2\cos\theta(2\sin\theta - \cos\theta) = 0$$

$$= 1 - 2\sin^2\theta$$

$$\cos\theta = 0^\circ \quad \text{or} \quad 2\sin\theta - \cos\theta = 0$$

$$= 2\cos^2\theta - 1$$

$$\theta = 90^\circ$$

$$2\sin\theta = \cos\theta$$

$$\frac{\sin\theta}{\cos\theta} = \frac{1}{2}$$

$$\tan\theta = \frac{1}{2}$$

$$\theta = \arctan\left(\frac{1}{2}\right)$$

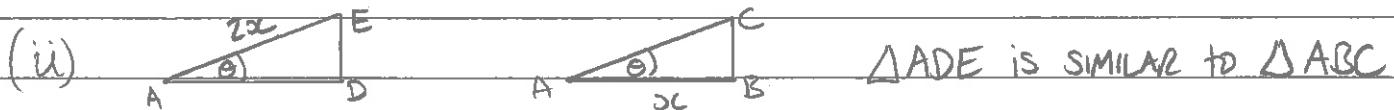
$$\text{values } 0^\circ \leq \theta \leq 180^\circ \text{ are } \underline{\underline{0.266^\circ}} \text{ (3s.f.)}, 90^\circ$$

$$5(i) \quad \cos \theta = \frac{x}{AC} \quad \Delta ABC \quad \text{using "SOHCAHTOA"} \\ \sec \theta = \frac{AC}{x} \\ \csc \theta = AC \quad ①$$

$$\cos \theta = \frac{AC}{AD} \quad \Delta ACD \\ \sec \theta = \frac{AD}{AC} \quad \text{subst. for } AC \text{ from } ① \\ x \sec^2 \theta = AD \quad ②$$

$$\cos \theta = \frac{AD}{2x} \quad \Delta ADE \\ \sec \theta = \frac{2x}{AD} \quad \text{subst. for } AD \text{ from } ② \\ \cancel{x} \sec^3 \theta = 2x$$

$$\therefore \underline{\underline{\sec^3 \theta = 2}} \quad \text{Q.E.D.}$$



\therefore ratio of $ED:CB$ is same as ratio $AE:AC$, which is

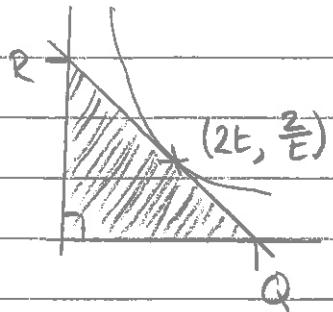
$$2x : x \sec \theta \quad (\text{using } ①) \\ 2 : 2^{1/3} \\ 2 : 2^{1/3} \\ \frac{2}{2^{1/3}} : \frac{2^{1/3}}{2^{1/3}} \\ 2^{2/3} : 1 \quad \text{QED}$$

$$\sec \theta = \sqrt[3]{2} \quad \text{from part (i)} \\ = 2^{1/3}$$

$$6. \quad x = 2t, \quad y = 2t^{-1}$$

$$\frac{dx}{dt} = 2, \quad \frac{dy}{dt} = -2t^{-2}$$

$$= -\frac{2}{t^2}$$



(find x- & y- intercepts and
use $\Delta \text{area} = \frac{1}{2}bh$)

$$\frac{dy}{dx} = -\frac{2/t^2}{2} = -\frac{1}{t^2}$$

$= -\frac{1}{t^2}$ = grad. of line segment RQ.

$$y - y_1 = m(x - x_1)$$

$$y - \frac{2}{t} = -\frac{1}{t^2}(x - 2t) \quad \text{is EQUATION of line RQ}$$

y-int. where $x=0$

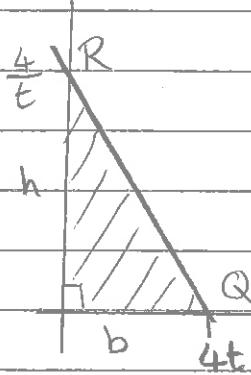
$$y - \frac{2}{t} = -\frac{1}{t^2}(-2t)$$

$$y = \frac{2t}{t^2} + \frac{2}{t}$$

$$= \frac{4}{t}$$

x-int. where $y=0$

$$-\frac{2}{t} = -\frac{1}{t^2}(x - 2t)$$



$$+\frac{2t^2}{t} = x - 2t$$

$$x = 4t$$

$$\Delta \text{OQR AREA} = \frac{1}{2}(4t)(\frac{4}{t})$$

$$\Delta \text{AREA} = \frac{1}{2}bh$$

$= 8$ square units,
which is INDEPENDENT
of t. Q.E.D.

$$7(i) |\vec{AG}| = |\underline{g} - \underline{a}|$$

$$= \left| \begin{pmatrix} 0 \\ 3 \\ 5 \end{pmatrix} - \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} \right|$$

$$= \left| \begin{pmatrix} -4 \\ 3 \\ 5 \end{pmatrix} \right|$$

$$= \sqrt{4^2 + 3^2 + 5^2}$$

$$= \sqrt{50}$$

$$= 5\sqrt{2}$$

(ii) Need to show \underline{n} is NORMAL to 2 different vectors in plane.

If $\underline{n} \cdot \underline{v} = 0$, \underline{n} is NORMAL to \underline{v}

plane DPF, using vectors \vec{DP} and \vec{DF}

$$\begin{aligned} \vec{DP} \cdot \underline{n} &= \begin{pmatrix} 4 \\ 2 \\ -5 \end{pmatrix} \cdot \begin{pmatrix} 15 \\ -20 \\ 4 \end{pmatrix} = -5(15) \\ &= 4(15) + 2(-20) + (-5)(4) \\ &= 60 - 40 - 20 \\ &= 0 \end{aligned}$$

$$\begin{aligned} \vec{DF} \cdot \underline{n} &= \begin{pmatrix} 4 \\ 3 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 15 \\ -20 \\ 4 \end{pmatrix} \\ &= 4(15) + 3(-20) + 0 \\ &= 60 - 60 \\ &= 0 \end{aligned}$$

$\therefore \underline{n}$ is NORMAL to plane DPF.

CARTESIAN EQUATION is $15x - 20y + 4z = d$

$$\begin{aligned} \text{subst. pt. } D(0,0,5) \quad 0 - 0 + 20 &= d \\ 20 &= d \end{aligned}$$

$\therefore \underline{15x - 20y + 4z = 20}$ is CART. EQN.

$$(iii) \text{ line AG is } \underline{r} = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -4 \\ 3 \\ 5 \end{pmatrix}$$

7(iii)
cont'd

Q is at intersection of line $\underline{r} = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -4 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

and plane $15x - 20y + 4z = 20$. Subst. line coords into plane:

$$15(4 - 4\lambda) - 20(3\lambda) + 4(5\lambda) = 20$$

$$60 - 60\lambda - 60\lambda + 20\lambda = 20$$

$$-100\lambda = -40$$

$$\lambda = \frac{2}{5} \text{ subst into line}$$

$$\begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} + \frac{2}{5} \begin{pmatrix} -4 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} \frac{12}{5} \\ \frac{6}{5} \\ 2 \end{pmatrix}$$

$\therefore Q$ is pt. $(\frac{12}{5}, \frac{6}{5}, 2)$ or $(2.4, 1.2, 2)$

ratio $AQ : QG$ is $\left| \begin{pmatrix} -1.6 \\ 1.2 \\ 2 \end{pmatrix} \right| : \left| \begin{pmatrix} -2.4 \\ 1.8 \\ 3 \end{pmatrix} \right|$
 $|g - g_1| : |g - g_2|$

$$\sqrt{1.6^2 + 1.2^2 + 2^2} : \sqrt{2.4^2 + 1.8^2 + 3^2}$$
$$2\sqrt{2} : 3\sqrt{2}$$

$$\therefore AQ : QG = \underline{2 : 3}$$

(iv) θ between line and plane is $90^\circ - (\theta$ between line & normal to plane)

θ between \underline{n} and direction of AG is $\Theta = \arccos \left(\frac{\underline{n} \cdot \overrightarrow{AG}}{|\underline{n}| |\overrightarrow{AG}|} \right)$

$$\underline{n} \cdot \overrightarrow{AG} = \begin{pmatrix} 15 \\ -20 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} -4 \\ 3 \\ 5 \end{pmatrix}$$

$$= -60 - 60 + 20$$

$$= -100$$

$$|\underline{n}| = \sqrt{15^2 + 20^2 + 4^2}$$
$$= \sqrt{641}$$

$$|\overrightarrow{AG}| = 5\sqrt{2}$$

$$\Theta = \arccos \left(\frac{-100}{5\sqrt{2}\sqrt{641}} \right)$$

$$= 123.957\dots$$

acute θ is $180 - 123.957\dots$

$$= 56.04 \text{ (4 s.f.)}$$

θ required is $90^\circ - 56.04^\circ = \underline{34.0^\circ}$ (3 s.f.)

$$\begin{aligned}
 8(i) \quad LHS &= \frac{1}{2+x} + \frac{1}{2-x} \\
 &= \frac{2-x}{(2+x)(2-x)} + \frac{2+x}{(2+x)(2-x)} \\
 &= \frac{4}{(2+x)(2-x)} \\
 &= RHS \qquad \text{QED}
 \end{aligned}$$

$$\begin{aligned}
 (ii) \quad t &= \ln\left(\frac{2+x}{2-x}\right) \\
 \text{when } t=0, \quad 0 &= \ln\left(\frac{2+x}{2-x}\right) \\
 e^0 &= \frac{2+x}{2-x} \qquad \qquad e^0 = 1
 \end{aligned}$$

$$\begin{aligned}
 2-x &= 2+x \\
 0 &= 2x \\
 x &= 0 \quad \therefore \text{when } t=0, x=0. \quad \text{QED}
 \end{aligned}$$

(iii) rate of change of x is $\frac{dx}{dt}$. First find $\frac{dt}{dx}$, then reciprocate.

$$\begin{aligned}
 t &= \ln(2+x) - \ln(2-x) \\
 \frac{dt}{dx} &= \frac{1}{2+x} + \frac{1}{2-x} \\
 &= \frac{4}{(2+x)(2-x)} \quad \text{from part(i)}
 \end{aligned}$$

$$\frac{dx}{dt} = \frac{1}{4}(2+x)(2-x) \quad \therefore \text{rate is proportional to} \\
 \text{given product with constant of} \\
 \text{proportionality } \underline{\frac{1}{4}}$$

$$(iv) t = \ln\left(\frac{2+x}{2-x}\right)$$

$$e^t = \frac{2+x}{2-x}$$

$$(2-x)e^t = 2+x$$

$$2e^t - xe^t = 2+x$$

$$2e^t - 2 = x + xe^t$$

$$2(e^t - 1) = x(1 + e^t)$$

$$x = \frac{2(e^t - 1)}{(e^t + 1)} \times \frac{e^{-t}}{e^{-t}}$$

$$= \frac{2(1 - e^{-t})}{(1 + e^{-t})} \quad Q.E.D.$$

As $t \rightarrow \infty$, $e^{-t} \rightarrow 0$, $x \rightarrow 2 \therefore$ long term mass is 2 mg

$$(v) \frac{dx}{dt} = k(2+x)(2-x)e^{-t} \quad \text{separate var.}$$

$$\int \frac{1}{(2+x)(2-x)} dx = k \int e^{-t} dt$$

$$\frac{1}{4} \int \frac{1}{2+x} + \frac{1}{2-x} dx = -k e^{-t} + C$$

$$\begin{aligned} \ln\left(\frac{2+x}{2-x}\right) &= -4k e^{-t} + C \\ &= -4k e^{-t} + 4k \\ &= 4k(1 - e^{-t}) \end{aligned}$$

subst $x=0, t=0$

to find C

$$\ln\left(\frac{2}{2}\right) = -4k e^0 + C$$

$$0 = -4k + C$$

$$4k = C$$

Q.E.D.

(vi) in the long term, $x = 1.85$, $e^{-t} \rightarrow 0$

$$\ln\left(\frac{2+1.85}{2-1.85}\right) = 4k$$

$$k = \frac{1}{4} \ln\left(\frac{77}{3}\right) = \underline{\underline{0.811}} \text{ (3.s.f.)}$$