C4 summer 2015

SECTION A

1 $\cancel{(2x+1)}\dfrac{5x}{\cancel{2x+1}}(x+1) - (2x+1)\dfrac{3}{\cancel{x+1}}\cancel{(x+1)} = 1(2x+1)(x+1)$

$5x^2 + 5x - 6x - 3 = 2x^2 + 3x + 1$

$3x^2 - 4x - 4 = 0$

$3x^2 - 6x + 2x - 4 = 0$

$3x(x-2) + 2(x-2) = 0$

$(3x+2)(x-2) = 0$

$x = -\frac{2}{3}, \quad x = 2$

2 $6\cos 2\theta + \sin\theta$

$= 6(1 - 2\sin^2\theta) + \sin\theta$

$= 6 - 12\sin^2\theta + \sin\theta$

$\cos 2\theta \equiv \cos^2\theta - \sin^2\theta$

$\equiv 1 - 2\sin^2\theta$

let $\sin\theta = s$

$12s^2 - s - 6 = 0$

$12s^2 + 8s - 9s - 6 = 0$

$4s(3s+2) - 3(3s+2) = 0$

$(4s-3)(3s+2) = 0$

$s = \frac{3}{4}, \quad s = -\frac{2}{3}$

$\arcsin\left(\frac{3}{4}\right) = 48.59\ldots$

$\arcsin\left(\frac{2}{3}\right) = 41.81\ldots$

$\therefore \theta = 48.6^\circ, 180^\circ - 48.6^\circ, 180^\circ + 41.8^\circ, 360^\circ - 41.8^\circ$ (3s.f.)

$\theta = 48.6^\circ, 131^\circ, 222^\circ, 318^\circ$ (3 s.f.)

3 (i) $\dfrac{1}{\sqrt[3]{1-2x}} = (1-2x)^{-1/3}$ $\quad n = -\frac{1}{3}$, use $(-2x)$

$\approx 1 + (-\tfrac{1}{3})(-2x) + \dfrac{(-\tfrac{1}{3})(-\tfrac{4}{3})(-2x)^2}{2} + \ldots$

$= 1 + \frac{2}{3}x + \frac{8}{9}x^2 + \ldots$

valid for $|-2x| < 1 \Rightarrow |x| < \frac{1}{2}$

(ii) $\dfrac{1-3x}{\sqrt[3]{1-2x}} \approx (1-3x)(1 + \frac{2}{3}x + \frac{8}{9}x^2 + \ldots)$

$= 1 + \frac{2}{3}x + \frac{8}{9}x^2 - 3x - 2x^2 - x^3$ term ...

$= 1 - \frac{7}{3}x - \frac{10}{9}x + \ldots$ $\quad$ valid for $|x| < \frac{1}{3}$

$\therefore a = -\frac{7}{3}, \; b = -\frac{10}{9}$

4 (i) $f(x) = \cos x + \lambda \sin x = R\cos(x - \alpha)$

$= R\cos x \cos\alpha + R \sin x \sin\alpha$

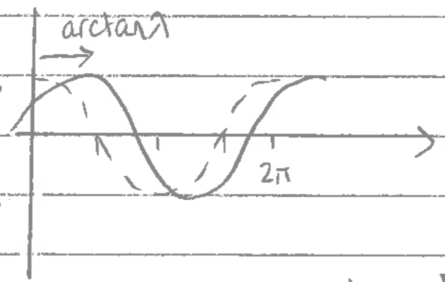
equate coeff.s $\cos x: \quad 1 = R\cos\alpha$

$\sin x: \quad \lambda = R\sin\alpha$

$\dfrac{\cancel{R}\sin\alpha}{\cancel{R}\cos\alpha} = \dfrac{\lambda}{1}$, $\quad \tan\alpha = \lambda$ $\quad\quad R = \sqrt{1+\lambda^2}$

$\alpha = \arctan\lambda$

$\therefore f(x) = \sqrt{1+\lambda^2}\left(\cos(x - \arctan\lambda)\right)$



$R = 2$ $\quad \sqrt{1+\lambda^2} = 2$

$1 + \lambda^2 = 4$

$\lambda^2 = 3$

$\lambda = \sqrt{3}$

$\alpha = \arctan\sqrt{3}$

$\alpha = \frac{\pi}{3}$

5 (i) $x = \sec\theta$, $y = 2\tan\theta$

$\frac{dx}{d\theta} = \sec\theta\tan\theta$, $\frac{dy}{d\theta} = 2\sec^2\theta$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{2\sec^{\cancel{2}}\theta}{\cancel{\sec\theta}\tan\theta} = \frac{2\cancel{\cos\theta}}{\cancel{\cos\theta}\sin\theta} = 2\operatorname{cosec}\theta \quad \text{Q.E.D.}$$

using $\sec\theta = \frac{1}{\cos\theta}$

$\tan\theta = \frac{\sin\theta}{\cos\theta}$

$\operatorname{cosec}\theta = \frac{1}{\sin\theta}$

(ii) $y^2 = 4\tan^2\theta$
$= 4(\sec^2\theta - 1)$
$= 4x^2 - 4$ QED

$\cos^2\theta + \sin^2\theta \equiv 1$
$1 + \tan^2\theta \equiv \sec^2\theta$
$\tan^2\theta \equiv \sec^2\theta - 1$

(iii) $\text{volume} = \pi\int_1^2 4x^2 - 4 \; dx$

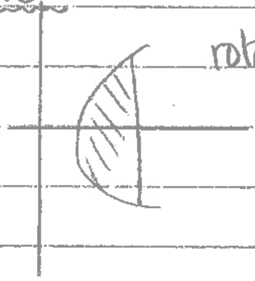
$= \pi\left[\frac{4}{3}x^3 - 4x\right]_1^2$

$= \pi\left(\left(\frac{4}{3}(2)^3 - 4(2)\right) - \left(\frac{4}{3} - 4\right)\right)$

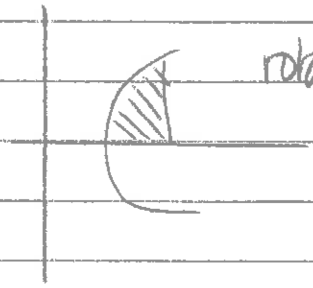
$= \pi\left(\frac{32}{3} - \frac{24}{3} - \left(\frac{4}{3} - \frac{12}{3}\right)\right)$

$= \underline{\underline{\frac{16}{3}\pi}}$

NOTE:



rotated 180° is same as



rotated 360°.

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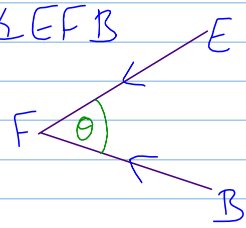
6(iii) $\angle EFB$



$$\vec{EF} = \begin{pmatrix} 2-0 \\ a--2 \\ 2-3 \end{pmatrix} = \begin{pmatrix} 2 \\ -\frac{1}{3} \\ -1 \end{pmatrix} \qquad \vec{BF} = \begin{pmatrix} 2-3 \\ a--2.5 \\ 2-0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1/6 \\ 2 \end{pmatrix}$$

$$|\vec{EF}| = \sqrt{2^2 + \left(\tfrac{1}{3}\right)^2 + 1^2} = \frac{\sqrt{46}}{3}$$

$$|\vec{BF}| = \sqrt{1^2 + \left(\tfrac{1}{6}\right)^2 + 2^2} = \frac{\sqrt{181}}{6}$$

$$\vec{EF} \cdot \vec{BF} = 2\times -1 + -\tfrac{1}{3} \times \tfrac{1}{6} + -1 \times 2 = -2 - \frac{1}{18} - 2 = -\frac{73}{18}$$

let $\angle EFB = \theta$, $\cos\theta = \dfrac{\vec{EF} \cdot \vec{BF}}{|\vec{EF}||\vec{BF}|}$

$$\theta = \arccos\left(\frac{-\frac{73}{18}}{\frac{\sqrt{46}}{3} \times \frac{\sqrt{181}}{6}}\right)$$

$$\angle EFB = 143.13\ldots$$

$$= \underline{143^\circ} \ (3\text{ s.f.})$$

(iv) Position vector of P is given by

$\vec{OP} = \vec{OF} + \frac{1}{2}\vec{FE}$ using $\vec{FE} = -\vec{EF}$ (found earlier)

$$= \begin{pmatrix} 2 \\ -7/3 \\ 2 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} -2 \\ \frac{1}{3} \\ 1 \end{pmatrix}$$

z-coord of P is 2.5 (z-coord is height above ground)

Position vector of $\vec{OQ}$ is given by

$$\vec{OQ} = \vec{OP} + \frac{1}{3}\vec{PH}$$

z coord of $q = 2.5 + \frac{1}{3}(3-2.5)$

$$= \frac{8}{3}$$

$\therefore$ Q is $\underline{2^{2}/_{3} \text{ metres}}$ above the ground.

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SECTION B

6(i) A $(0,-2,0)$ subst. A, B, E into LHS of eqn of plane, should get 0.

B $(3,-2.5,0)$

E $(0,-2,3)$

A: $0+6(-2)+12$
$=0-12+12$
$=0$

B: $3+6(-2.5)+12$
$=3-15+12$
$=0$

E: $0+6(-2)+12$
$=0-12+12$
$=0$

$\therefore$ A, B, E all lie on the plane $x+6y+12=0$

so $x+6y+12=0$ is the eqn. of the plane thro' A, B & E.

F$(2,a,2)$ lies in plane $\therefore$

$$2+6a+12=0$$
$$6a=-14$$
$$a=-\frac{14}{6}$$
$$=-\frac{7}{3}$$
$$=-2\tfrac{1}{3} \quad \text{QED}$$

(ii) (A) We require $\cdot$ of $\begin{pmatrix}1\\-6\\0\end{pmatrix}$ and 2 different vectors in the plane to be zero.

DH lies in the plane $\quad \overrightarrow{DH}=\begin{pmatrix}0\\1\\3\end{pmatrix}-\begin{pmatrix}0\\1\\0\end{pmatrix}=\begin{pmatrix}0\\0\\3\end{pmatrix}$

DC lies in the plane. $\quad \overrightarrow{DC}=\begin{pmatrix}3\\1.5\\0\end{pmatrix}-\begin{pmatrix}0\\1\\0\end{pmatrix}=\begin{pmatrix}3\\0.5\\0\end{pmatrix}$

$$\overrightarrow{DH}\cdot\begin{pmatrix}1\\-6\\0\end{pmatrix}=\begin{pmatrix}0\\0\\3\end{pmatrix}\cdot\begin{pmatrix}1\\-6\\0\end{pmatrix}$$
$$=0\times1+0\times-6+3\times0$$
$$=0$$

$$\overrightarrow{DC}\cdot\begin{pmatrix}1\\-6\\0\end{pmatrix}=\begin{pmatrix}3\\0.5\\0\end{pmatrix}\cdot\begin{pmatrix}1\\-6\\0\end{pmatrix}$$
$$=3\times1+0.5\times-6+0\times0$$
$$=0$$

$\therefore \begin{pmatrix}1\\-6\\0\end{pmatrix}$ is $\perp$ to the plane DHC

(B) D lies on the plane, subst. into

$$x-6y+0z=d$$
$$0-6(1)+0=d$$
$$-6=d$$

$\therefore \underline{x-6y+6=0}$ is cartesian eqn. of this plane.

(C) subst. G$(2,b,2)$ into eqn. of plane

$$2-6b+6=0$$
$$-6b=-8$$
$$\underline{b=\frac{4}{3}}$$

$$|\overrightarrow{FG}|=\sqrt{(2-2)^2+(\tfrac{4}{3}--2\tfrac{1}{3})^2+(2-2)^2}$$
$$=\frac{11}{3}$$
$$=\underline{3\tfrac{2}{3}\text{ metres}}$$

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7(i) $\dfrac{1}{(1+2x)(1-x)} \equiv \dfrac{A^{(1-x)}}{(1+2x)_{(1-x)}} + \dfrac{B^{(1+2x)}}{(1-x)_{(1+2x)}} \equiv \dfrac{A(1-x)+B(1+2x)}{(1+2x)(1-x)}$

equate numerators: $1 = A(1-x) + B(1+2x)$

$1 = A - Ax + B + 2Bx$

subst. $x=1$: $1 = B(1+2(1))$

$\frac{1}{3} = B$

equate const.: $1 = A + B$

$A = \frac{2}{3}$

$\therefore \dfrac{1}{(1+2x)(1-x)} \equiv \dfrac{2}{3(1+2x)} + \dfrac{1}{3(1-x)}$

(ii) $\dfrac{dx}{dt} = k(1+x-2x^2)$

$\dfrac{dx}{dt} = k(1+2x)(1-x)$

$\displaystyle\int \frac{1}{(1+2x)(1-x)}\,dx = \int k\,dt$

$\displaystyle\frac{1}{3}\int \frac{2}{1+2x} + \frac{1}{1-x}\,dx = \int k\,dt$

$\ln|1+2x| - \ln|1-x| = 3kt + c$

$\ln\left|\dfrac{1+2x}{1-x}\right| = 3kt + c$

$\dfrac{d}{dx}\left(-\ln|1-x|\right) = \dfrac{-1}{1-x} \times -1$

subst. $x=0$, $t=0$

$\ln 1 = 0 + c$

$0 = c$

$\ln\left|\dfrac{1+2x}{1-x}\right| = 3kt$

$\dfrac{1+2x}{1-x} = e^{3kt}$ QED

(iii) subst. $t=1$, $x=0.75$

$\dfrac{1+2(0.75)}{1-0.75} = e^{3k}$

$\dfrac{1+2\left(\frac{3}{4}\right)}{1-\frac{3}{4}} = \dfrac{\frac{4+6}{4}}{\frac{1}{4}} = 10$

$3k = \ln 10$

$k = \frac{1}{3}\ln 10$

$= 0.768$ (3s.f.)

subst. $x = 0.9$

$\dfrac{1+2(0.9)}{1-0.9} = e^{\ln 10\, t}$

$e^{\ln 10} = 10$

$28 = 10^t$

$e^{\ln 10 t} = (e^{\ln 10})^t$

$t = \log_{10} 28$

$= 1.4471\ldots$

$= 1.45$ hours (3s.f.)

(ii) $\dfrac{1+2x}{1-x} = e^{3kt}$

$1+2x = e^{3kt}(1-x)$

$1+2x = e^{3kt} - xe^{3kt}$

$2x + xe^{3kt} = e^{3kt} - 1$

$x(2+e^{3kt}) = e^{3kt} - 1$

$x = \dfrac{e^{3kt}-1}{2+e^{3kt}} \times \dfrac{e^{-3kt}}{e^{-3kt}}$

$= \dfrac{1-e^{-3kt}}{2e^{-3kt}+1}$ QED

In the long term, as $t \to \infty$, $e^{-3kt} \to 0$

$\therefore x \to \dfrac{1-0}{0+1} = 1$, which is the maximum value.