

C4 Summer 2014

* Learn the three forms of partial fractions you may need to use

$$\frac{3x}{(2-x)(4+x^2)} * = \frac{A}{2-x} + \frac{Bx+C}{4+x^2} \equiv \frac{A(4+x^2)}{4+x^2} + \frac{(Bx+C)(2-x)}{4+x^2}$$

constant numerator linear numerator
linear factor quadratic factor

equate num: $3x = A(4+x^2) + (Bx+C)(2-x)$

$$3x = 4A + Ax^2 + 2Bx - Bx^2 + 2C - Cx$$

subst. $x=2$:

$$3(2) = A(4+2^2) + 0$$

$$6 = 8A$$

$$\frac{3}{4} = A$$

take care ~ lots of brackets and -ves to deal with !!

equate const. terms:

$$0 = 4A + 2C$$

$$0 = 3 + 2C$$

$$-\frac{3}{2} = C$$

equate x^2 terms:

$$0 = A - B$$

$$0 = \frac{3}{4} - B$$

$$B = \frac{3}{4}$$

common denom.

$$\therefore \frac{3x}{(2-x)(4+x^2)} \equiv \frac{3}{4(2-x)} + \frac{3x-6}{4(4+x^2)} \quad C = -\frac{6}{4}$$

$$\begin{aligned} 2 \quad (4+x)^{3/2} &= (4(1+\frac{x}{4}))^{3/2} \\ &= 8(1+\frac{x}{4})^{3/2} \\ &= 8\left(1 + (\frac{3}{2})(\frac{x}{4}) + \frac{(\frac{3}{2})(\frac{1}{2})}{2!}\left(\frac{x}{4}\right)^2 + \dots\right) \end{aligned}$$

make sure you can deal with the "4" to leave $(1+\dots)^n$

$$= 8\left(1 + \frac{3x}{8} + \frac{3x^2}{128} + \dots\right)$$

take extra care with brackets !!

$$= 8 + 3x + \frac{3}{16}x^2 + \dots$$

valid for $|x| < 1$

don't forget to include the validity

STRICT inequality

$<$ not \leq

$$|x| < 4$$

↑ could alternatively be written

$$-4 < x < 4$$

3(i)	x	0	$\frac{\pi}{16}$	$\frac{\pi}{8}$	$\frac{3\pi}{16}$	$\frac{\pi}{4}$	$h = \frac{\pi/4}{4} = \frac{\pi}{16}$
		0	0.4493	0.6792	0.9698	1.3254	

$$\text{AREA} \approx \frac{1}{2} \left(\frac{\pi}{16} \right) (0 + 1.3254 + 2(0.4493 + 0.6792 + 0.9698))$$

↙ 3d.p. required, so values of y must be found to AT LEAST 4 d.p.

$$= 0.538 \text{ sq. units (3d.p.)}$$

(ii) Not possible to say, as curve has both concave & convex parts

$$4 \quad (i) \quad \text{RHS} = 1 - \frac{\tan \alpha \tan \beta}{\sec \alpha \sec \beta}$$

$$= 1 - \frac{\frac{\sin \alpha}{\cos \alpha} \cdot \frac{\sin \beta}{\cos \beta}}{\frac{1}{\cos \alpha} \cdot \frac{1}{\cos \beta}}$$

$$= \cos \alpha \cos \beta - \cos \alpha \cos \beta \left(\frac{\sin \alpha \cdot \sin \beta}{\cos \alpha \cos \beta} \right)$$

$$= \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

$$= \cos(\alpha + \beta) \quad \text{by compound angle formula}$$

$$= \text{LHS} \quad \text{Q.E.D.}$$

using $\sec \alpha = \frac{1}{\cos \alpha}$

and $\tan \alpha = \frac{\sin \alpha}{\cos \alpha}$

show clear steps leading from one side of the equation to the other. You are not finding an "answer" ~ credit is given for showing the working!!

(ii) let $\alpha = \beta$ in identity from (i), then

$$\text{L.H.S.} = \cos 2\alpha = \cos(\alpha + \alpha)$$

$$= \frac{1 - \tan^2 \alpha}{\sec^2 \alpha}$$

using IDENTITY $\sin^2 \alpha + \cos^2 \alpha = 1$

$$\tan^2 \alpha + 1 = \sec^2 \alpha$$

$$= \frac{1 - \tan^2 \alpha}{1 + \tan^2 \alpha}$$

Q.E.D.

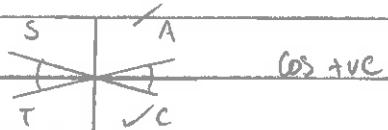
$$= \text{R.H.S.}$$

(iii) from (ii), $\cos 2\alpha = \frac{1}{2}$

$$2\alpha = 60^\circ, 300^\circ$$

$$\alpha = 30^\circ, 150^\circ$$

make sure you find all values in given range



y is the product of functions t and e^{2t}

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$$x = e^{3t}$$

$$\frac{dx}{dt} = 3e^{3t}$$

$$y = te^{2t} \quad \text{0 0 0 0}$$

$$\frac{dy}{dt} = 2te^{2t} + e^{2t}$$

PRODUCT RULE

$$u = t$$

$$v = e^{2t}$$

$$u' = 1$$

$$v' = 2e^{2t}$$

$$= (2t+1)e^{2t}$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{e^{2t}(2t+1)}{3e^{3t}} \\ &= \frac{2t+1}{3e^t}\end{aligned}$$

when $t = 1$

$$\text{grad is } \frac{2(1)+1}{3e^1}$$

$$= \frac{3}{3e}$$

$$= \frac{1}{e}$$

(ii) $x = e^{3t}$

$$\ln x = 3t$$

$$\frac{1}{3} \ln x = t$$

Subst. into $y = te^{2t}$

$$\begin{aligned}&= \frac{1}{3} \ln x \times e^{2 \times \frac{1}{3} \ln x} \\ &= \frac{1}{3} \ln x \times e^{\ln x^{\frac{2}{3}}}\end{aligned}$$

using $e^{\ln k} = k$

because e and \ln are inverse functions

$$y = \frac{1}{3} x^{\frac{2}{3}} \ln x$$

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$$y = (1 + 2x^2)^{\frac{1}{3}}$$

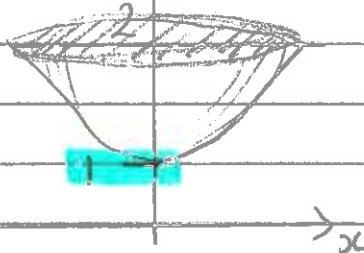
$$\text{y-int, } x=0 \quad y = (1+0)^{\frac{1}{3}}$$

$$= \sqrt[3]{1}$$

RTQ ~ rotate around y -axis, so

need $f \dots dy$

\therefore need to find x^2 in terms of y



need to find lower limit

$$\text{area} = \pi \int_{-1}^2 x^2 dy$$

$$= \frac{1}{2} \pi \int_{-1}^2 y^3 - 1 dy$$

$$= \frac{1}{2} \pi [\frac{1}{4} y^4 - y]_1^2$$

$$= \frac{1}{2} \pi \left((\frac{1}{4}(2)^4 - 2) - (\frac{1}{4}(1) - 1) \right)$$

$$= \frac{1}{2} \pi \left(2 - -\frac{3}{4} \right)$$

$$= \frac{11}{8} \pi \text{ cubic units}$$

$$y^3 = 1 + 2x^2$$

$$\frac{1}{2}(y^3 - 1) = x^2$$

$$7(i) |\vec{AB}| = \sqrt{5^2 + 0^2 + 2^2} = \underline{\underline{\sqrt{29}}}$$

$$|\vec{AC}| = \sqrt{3^2 + 4^2 + 0^2} = \underline{\underline{5}}$$

$$\vec{AB} = \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix} - \begin{pmatrix} -3 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ -2 \end{pmatrix}$$

$$\vec{AC} = \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix} - \begin{pmatrix} -3 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}$$

$$\vec{AB} \cdot \vec{AC} = 5(3) + 0 + 0 = 15$$

using clear notation helps
keep track of what you
have found

$$\angle CAB = \arccos \left(\frac{\vec{AB} \cdot \vec{AC}}{|\vec{AB}| |\vec{AC}|} \right)$$

$$= \arccos \left(\frac{15}{5\sqrt{29}} \right)$$

$$= \underline{\underline{56.1^\circ}} \quad (3 \text{ s.f.})$$

$$\text{AREA } \Delta ABC = \frac{1}{2} |\vec{AB}| |\vec{AC}| \sin(\angle CAB)$$

$$= \frac{1}{2} \sqrt{29}(5) \sin(\arccos \frac{15}{5\sqrt{29}})$$

need to show two different vectors in the plane are \perp to the normal
 \perp to the normal

(ii) need to show scalar product of $\begin{pmatrix} 4 \\ -3 \\ 10 \end{pmatrix}$ and \vec{AB} or \vec{AC} gives zero.

(A)

$$\begin{pmatrix} 4 \\ -3 \\ 10 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 0 \\ -2 \end{pmatrix} = 20 + 0 - 20 = 0$$

$\therefore \perp$ to \vec{AB}

$$\begin{pmatrix} 4 \\ -3 \\ 10 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix} = 12 - 12 + 0 = 0$$

$\therefore \perp$ to \vec{AC}

\therefore the given vector is normal to plane ABC

(b) equation of plane is $4x - 3y + 10z = a \cdot n$

$$a \cdot n = \begin{pmatrix} -3 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ -3 \\ 10 \end{pmatrix} = -12$$

$$\therefore 4x - 3y + 10z + 12 = 0$$

must have $\underline{\underline{1}}$ or $\underline{\underline{\frac{y}{z}}}$ on LHS of line equation

$$(iii) \underline{\underline{\lambda = \begin{pmatrix} 0 \\ 4 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -3 \\ 10 \end{pmatrix}}} \quad \text{subst line into plane eqn:}$$

$$4(4\lambda) - 3(4-3\lambda) + 10(5+10\lambda) + 12 = 0$$

$$16\lambda - 12 + 9\lambda + 50 + 100\lambda + 12 = 0$$

$$125\lambda = -50$$

$$\therefore \text{pt. of intersection } \begin{pmatrix} 0 \\ 4 \\ 5 \end{pmatrix} - \frac{2}{5} \begin{pmatrix} 4 \\ -3 \\ 10 \end{pmatrix}, \text{ so pt. } \left(\frac{-8}{5}, \frac{26}{5}, 1 \right) \quad \lambda = \underline{\underline{-\frac{2}{5}}}$$

height of tetrahedron is length between D and pt. found in (iii)

$$h = \sqrt{\left(\frac{8}{5}\right)^2 + \left(\frac{6}{5}\right)^2 + 4^2} \quad \therefore \text{volume} = \frac{1}{3} (11.2) \sqrt{20}$$

this is the distance from D to the plane

note that the base of the tetrahedron is not just the xy-plane.

$$= \underline{\underline{16.7}} \text{ cubic units (3 s.f.)}$$

"verify" so we don't need to integrate ~ can just show result fits by DIFFERENTIATION

8 (i) $h = (1 - \frac{1}{2}At)^2 \Rightarrow \sqrt{h} = 1 - \frac{1}{2}At$

$\frac{dh}{dt} = 2(1 - \frac{1}{2}At) \times -\frac{1}{2}A$ by the CHAIN rule
AND when $t=0, h=(1-0)^2=1$
 $= -A\sqrt{h}$ Q.E.D. as required

(ii) $h=0$ when $t=20 \Rightarrow$ $0 = (1 - \frac{1}{2}A(20))^2$ $= (1 - 10A)^2$ asked to find A, so make it clear what A is!! $A = \frac{1}{10}$	when $h=0.5$ $0.5 = (1 - \frac{1}{20}t)^2$ $\sqrt{0.5} = 1 - \frac{1}{20}t$ $t = (1 - \sqrt{0.5}) \times 20$ time = <u>5.86 s</u> (3.s.f.)
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(iii) $\frac{dh}{dt} = -B \frac{\sqrt{h}}{(1+h)^2}$ LHS of integral is $\int h^{1/2}(1+2h+h^2) dh = h^{-1/2} + 2h^{1/2} + h^{3/2}$

$\int \frac{(1+h)^2 dh}{\sqrt{h}} = \int -B dt$ ↗ you must include ∫ signs when you separate variables.

$\int h^{-1/2} + 2h^{1/2} + h^{3/2} dh = \int -B dt$ subst. $t=0 h=1$

$2h^{1/2} + \frac{4}{3}h^{3/2} + \frac{2}{5}h^{5/2} = -Bt + C$ $2 + \frac{4}{3} + \frac{2}{5} = C$

$h^{1/2}(2 + \frac{4}{3}h + \frac{2}{5}h^2) = \frac{56}{15} - Bt$ $C = \frac{30}{15} + \frac{20}{15} + \frac{6}{15}$
 $\downarrow \times 15$
 $\Rightarrow h^{1/2}(30 + 20h + 6h^2) = 56 - 15Bt$ Q.E.D.

(iv) subst $h=0, t=20$: $0 = 56 - 15(20)B$

$B = \frac{56}{15(20)}$ ↗ show clearly what B is
 $= \frac{14}{75} = 0.187$ (3.s.f.)

when $h=0.5$, $t = \frac{1}{15B} (56 - \sqrt{0.5}(30 + 10 + 6(0.25)))$
 $= \underline{\underline{9.52 s}}$ (3.s.f.)