C4 June 2010 worked solutions

Q1 Simple algebraic fraction, however you needed to note that the first denominator is THE LOWEST COMMON DENOMINATOR of both! Indeed $x^2 - 1 = (x-1)(x+1)$

Hence $\frac{x}{x^2-1} + \frac{2}{(x+1)} = \frac{x}{(x-1)(x+1)} + \frac{2(x-1)}{(x-1)(x+1)} = \boxed{\frac{3x-2}{(x-1)(x+1)}}$

Q2 i) Use of trapezium rule on $y = \sqrt{1+x^2}$ only one value to evaluate!

From table we see that h=0.25 and n=4strips so i=0 to 4

$$x = 0.5 \quad y = \sqrt{1+0.5^2} = \sqrt{1.25} \rightarrow \boxed{y = 1.1180}$$

Trapezium rule $\int_0^1 y\,dx = \frac{1}{2}h[y_0 + y_4 + 2(y_1+y_2+y_3)]$

$\int_0^1 \sqrt{1+x^2}\,dx = \frac{1}{2} \times 0.25 \times [1 + 1.4142 + 2(1.0308 + 1.1180 + 1.25)]$=1.151475

Which agrees to 3 d.p. with the area given of 1.151

ii) The curve is concave, the <u>trapezium rule will be an overestimate</u> (each trapezium will have top above the curve). As <u>we increase strips</u>, the results should be <u>more accurate</u> and therefore <u>overestimate less</u>. The value at 8 strips should be less than that for 4 strips. Hence her results being greater than the 4 strip value it must be wrong.

iii) This now becomes a volume integral, and therefore we will integrate $y = \sqrt{1+x^2}$ squared and no need for trapezium rule.

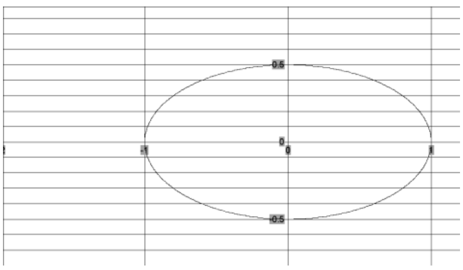
Volume=$\pi \int_0^1 y^2 dx = \pi \int_0^1 \left[\sqrt{1+x^2}\right]^2 dx = \pi \int_0^1 [1+x^2]^2 dx = \pi \left[x + \frac{x^3}{3}\right]_0^1 = \pi\left[1+\frac{1}{3}\right]$

$$\text{Volume } \boxed{\frac{4\pi}{3}}$$

Q3 For these type of question you need to check if some of trig relations can be used. The trig function seem to depend on either θ or 2θ. Try to express them in similar manner and then you can use $\sin^2\alpha + \cos^2\alpha = 1$. Then thinking of that you will see that $y = sin\theta cos\theta$ is close to double angle for sin!

$So\ using$ $y = \frac{1}{2} \times 2sin\theta cos\theta = \frac{1}{2} \times sin2\theta$ and we have $x = cos2\theta$

Now using $\sin^2 2\theta + \cos^2 2\theta = 1$ we get $(2y)^2 + x^2 = 1$ and $4y^2 + x^2 = 1$ as required



<u>An ellipse</u> goes through 1 and -1 on x-axis;1/2;-1/2 on y-axis. Note: either you know standard eq. of ellipse or you put in values, look at intersection with axis and use symmetry. !!it is not a circle, does certainly not look good if you just draw a circle.

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Q4 For these type of expansion first thing to do is 'make similar' to $(1 + something)^n$, hence we need to pull out factor 4 from square root $\sqrt{4+x} = \sqrt{4(1+\frac{x}{4})} = 2\sqrt{1+\frac{x}{4}} = 2(1+\frac{x}{4})^{\frac{1}{2}}$

Hence using $(1+x)^n = 1 + \frac{nx}{1!} + \frac{n(n-1)x^2}{2!} + \cdots$

$$\sqrt{4+x} \approx 2\left[1 + \frac{\frac{1}{2}}{1!}\frac{x}{4} + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{x}{4}\right)^2}{2!} + \cdots\right] = 2\left[1 + \frac{x}{8} + \frac{-1}{8}\frac{x^2}{16} + \cdots\right]$$

$\boxed{\sqrt{4+x} \approx 2 + \frac{x}{4} - \frac{x^2}{64} + \cdots}$ Valid for $\left|\frac{x}{4}\right| < 1 \quad \to |x| < 4$

Q5 Standard partial fraction question but if you read the question fully you will notice that the term $(y-2)(y+1)$ appears in the partial fraction and also in differential equation, hinting rather obviously that you will be using part i) to solve part ii) and also from all the work on partial fraction that ln's will be involved! i)

$$\frac{3}{(y-2)(y+1)} \equiv \frac{A}{y-2} + \frac{B}{y+1} = \frac{A(y+1)+B(y-2)}{(y-2)(y+1)}$$

$$3 \equiv A(y+1) + B(y-2)$$

Put $y = 2$ gives $3 = 3A \to \boxed{A = 1}$ and $y = 1$ $3 = -3B \to \boxed{B = -1}$

So that $\boxed{\frac{3}{(y-2)(y+1)} \equiv \frac{1}{y-2} - \frac{1}{y+1}}$

ii) We are given $\frac{dy}{dx} = x^2(y-2)(y+1)$. Need to separate variables (y's & dy LHS, x's and dx RHS) and integrate at the same time

$$\int \frac{1}{(y-2)(y+1)}dy = \int x^2 dx$$

NOTE LHS is same as LHS in part i) but divided by 3.

$$\frac{1}{3}\int \frac{3}{(y-2)(y+1)}dy = \int x^2 dx$$

$$\frac{1}{3}\int \left(\frac{1}{y-2} - \frac{1}{y+1}\right)dy = \int x^2 dx \quad \to \quad \frac{1}{3}(\ln(y-2) - \ln(y+1)) = \frac{x^3}{3} + C$$

Cancelling 1/3 and merging the ln's $\ln\left(\frac{y-2}{y+1}\right) = x^3 + C$

And now 'exponentiating' $\frac{y-2}{y+1} = \mathrm{e}^{x^3+C} = \mathrm{e}^{x^3}\mathrm{e}^C$

putting $\mathrm{A} = \mathrm{e}^C$, <u>it is very important to state this</u>, we get

$$\boxed{\frac{y-2}{y+1} = \mathrm{A}\mathrm{e}^{x^3}}$$

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Q6 Trigonometric equation which we need to express in terms of $\tan\theta$ and therefore need to change $\tan(\theta+45)$ using the compound formula replacing $\tan 45$ with 1 as soon as possible to make life easy!

$$\tan(\theta+45) = 1-2\tan\theta$$

$$\frac{\tan\theta+\tan 45}{1-\tan\theta\times\tan 45} = 1-2\tan\theta$$

$$\frac{\tan\theta+1}{1-\tan\theta} = 1-2\tan\theta$$

$$\tan\theta+1 = (1-2\tan\theta)(1-\tan\theta)$$

$$\tan\theta+1 = 1+2\tan^2\theta-3\tan\theta$$

$$2\tan^2\theta-4\tan\theta = 0$$ Note common factor $2\tan\theta$

$$2\tan\theta(\tan\theta-2) = 0$$

$\boxed{2\tan\theta = 0 \quad or \quad (\tan\theta-2)=0}$ Hence $\boxed{\theta = 0}$ or $\boxed{\theta = 63.43}$ for values between 0 and 90

Q7 A vector question where the first part is really on definitions. For second part, you could answer it without having done part i) as you are given the vector AB and a given point which gives you the vector eq. and you know that the vertical vector is that in the z direction i.e. $\vec{k}$ or $\begin{pmatrix}0\\0\\1\end{pmatrix}$, so only need to apply scalar product. Third part you need to remember that the intersection point is that whose *x*, *y* and *z* satisfy both the vector eq. of the line and the scalar eq. of the plane. For the last part remember that the normal vector has same component as coeff of scalar eq. Also will need to thing about which is the angle we need and which one will our scalar product give. i)

$$\overrightarrow{OB} = \begin{pmatrix}100\\200\\100\end{pmatrix} \qquad and \quad \overrightarrow{OA} = \begin{pmatrix}-200\\100\\0\end{pmatrix}$$

$$\overrightarrow{AB} = \overrightarrow{OB}-\overrightarrow{OA} = \begin{pmatrix}100\\200\\100\end{pmatrix} - \begin{pmatrix}-200\\100\\0\end{pmatrix} \rightarrow \boxed{\overrightarrow{AB} = \begin{pmatrix}300\\100\\100\end{pmatrix}}$$

Length is that of the vector $AB = \sqrt{300^2+100^2+100^2} = 100\sqrt{9+1+1} = 100\sqrt{11}$ <u>AB = 332m</u>

ii) Remember the vector eq. of a line is the position vector of a given point i.e. B, plus a parameter times a direction vector $\overrightarrow{AB}$ in our case $\boxed{\vec{r} = \overrightarrow{OB}+\lambda\overrightarrow{AB} = \begin{pmatrix}100\\200\\100\end{pmatrix}+\lambda\begin{pmatrix}300\\100\\100\end{pmatrix}}$ Note $\begin{pmatrix}3\\1\\1\end{pmatrix}$ could also be dir. vector.

Now the vertical vector is. $\vec{k}$ or $\begin{pmatrix}0\\0\\1\end{pmatrix}$ so we need $\cos\theta = \frac{\vec{k}\cdot\overrightarrow{AB}}{|\vec{k}|\cdot|\overrightarrow{AB}|}$

Now as $\vec{k}\cdot\overrightarrow{AB} = 0\times 300 + 0\times 100 + 1\times 100 = 100$ and $|\vec{k}| = 1$ by definition of basis vectors.

Need $$\cos\theta = \frac{100}{1\times 100\sqrt{11}} \quad\rightarrow \boxed{\theta = 72.45°}$$

iii) The vector equation of the line AB we had in part ii) needs to be expressed in parametric form.

$$\vec{r} = \begin{pmatrix}100\\200\\100\end{pmatrix} + \lambda\begin{pmatrix}300\\100\\100\end{pmatrix} \rightarrow \begin{matrix}x = 100+300\lambda\\ y = 200+100\lambda\\ z = 100+100\lambda\end{matrix}$$

These need to satisfy the equation of the plane π: $x + 2y + 3z = 320$, so that replacing *x*,*y* &*z*'s

$$(100+300\lambda) + 2(200+100\lambda) + 3(100+100\lambda) = 320$$

$$800\lambda = 320 - 800 \rightarrow \lambda = \frac{-480}{800} = -0.6$$

$$\begin{matrix}x = 100 - 180 = -80\\ y = 200 - 60 = 140\\ z = 100 - 60 = 40\end{matrix}$$

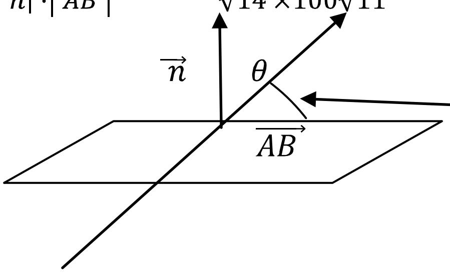
<u>point of intersection is **(-80,140,40)**</u>

iv) Now the normal to the plane is given by coeff of scalar eq. of plane π: $\mathbf{1}x + \mathbf{2}y + \mathbf{3}z = 320$

$$\vec{n} = \begin{pmatrix}1\\2\\3\end{pmatrix} \text{ and } |\vec{n}| = \sqrt{1+4+9} = \sqrt{14}$$

To find the angle between the plane and the pipeline we first evaluate the angle between pipeline, using the direction vector of the eq. of the pipeline, and the normal to the plane using the scalar product.

$$\cos\theta = \frac{\vec{n}\cdot\overrightarrow{AB}}{|\vec{n}|\cdot|\overrightarrow{AB}|} = \frac{1\times300+2\times100+3\times100}{\sqrt{14}\times100\sqrt{11}} = \frac{800}{\sqrt{14}\times100\sqrt{11}} = \frac{8}{\sqrt{14}\sqrt{11}} \quad \rightarrow \theta = 49.8$$



From diagram we see that the answer we need is

$$90 - \theta = 90 - 49.8 = \boxed{40.14°}$$

Q8 If you look at the curve then clearly A is when y=0 and B is when y has it lowest value, and as $y = 4cos\theta$ this can only occur when $cos\theta = -1$,its lowest possible value . Note that as B is a the min. we could also differentiate.

i) Given $x = 2\theta - sin\theta \quad y = 4cos\theta \qquad for\ 0 \le \theta \le 2\pi$

As A is the first root $y = 0 \to 4cos\theta = 0 \to \boxed{\theta = \frac{\pi}{2}}\ at\ A$

The most negative value for y is when $cos\theta = -1 \to \boxed{\theta = \pi}\ \ at\ B$

Now C is at the x-coordinate of B, i.e we need to evaluate $x = 2\theta - sin\theta$ at $\theta = \pi$

So at C $x = 2\pi - sin\pi$ $\boxed{x = 2\pi}$ at C

To dind the x-coordinate of A, i.e we need to evaluate $x = 2\theta - sin\theta$ at $\theta = \frac{\pi}{2}$

So at A $x = 2\frac{\pi}{2} - sin\frac{\pi}{2}$ $\boxed{x = \pi - 1}$ at A

Hence the segment OA = $\pi - 1$ and AC = OC-OA=$2\pi - (\pi - 1\ ) = \pi + 1$

Therefore we have shown $\boxed{OA: AC\ \ is\ (\pi - 1): (\pi + 1)}$

ii) Remember $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{-4\sin\theta}{2 - \cos\theta}$

Now at A we have $\theta = \frac{\pi}{2}$ so that $\frac{dy}{dx} = \frac{-4\sin\frac{\pi}{2}}{2 - \cos\frac{\pi}{2}} = \frac{-4}{2} \to \boxed{\frac{dy}{dx} = -2}$ at A

iii) $\frac{dy}{dx} = 1\ then\ \ \frac{-4\sin\theta}{2 - \cos\theta} = 1 \ \ \to -4\sin\theta \ = 2 - \cos\theta$

$\boxed{\cos\theta - 4\sin\theta \ = 2}$ as required.

iv) Need to remember that $\mathrm{R}\cos(\theta+\alpha)=\mathrm{a}\cos\theta-b\sin\theta$ be careful with the sign!

Hence in our problem a=1 and b=4 using $\mathrm{R}=\sqrt{a^2+b^2}$ and $\tan(\alpha)=\frac{\mathrm{b}}{\mathrm{a}}$

We get $\mathrm{R}=\sqrt{17}$ and $\alpha=\tan^{-1}\left(\frac{\mathrm{b}}{\mathrm{a}}\right)$ =1.326 Note important to use radians as this is what we have on our x-axis.

Hence $\boxed{\cos\theta-4\sin\theta=\sqrt{17}\cos(\theta+1.1326)}$

Therefore to solve $\cos\theta-4\sin\theta=2$

We can solve instead the easier eq. $\sqrt{17}\cos(\theta+1.1326)=2$

$$\cos(\theta+1.1326)=\frac{2}{\sqrt{17}}\rightarrow\theta+1.1326=1.604,\qquad 5.219,\qquad 7.348$$

It is important at this point to have included all extra sol, $2\pi-answer$ But also note that the values of possible sol is $0\leq\theta\leq2\pi$ this if for θ not $\theta+1.1326$ so you need at this stage the value of 7.348 (greater than 2π) as when we subtract 1.1326 we get back to possible values and $\theta=-0.262,\ 3.89,\ 6.02$

First value -0.262 out of range, so we get $\boxed{\theta=3.89,\ 6.02}$