Senior Mathematical Challenge

Thursday 7 November 2013

Organised by the United Kingdom Mathematics Trust

supported by Institute and Faculty of Actuaries

Solutions and investigations

These solutions augment the printed solutions that we send to schools. For convenience, the solutions sent to schools are confined to two sides of A4 paper and therefore in many cases are rather short. The solutions given here have been extended. In some cases we give alternative solutions, and we have included some exercises for further investigation.

The Senior Mathematical Challenge (SMC) is a multiple-choice paper. For each question, you are presented with five options, of which just one is correct. It follows that often you can find the correct answers by working backwards from the given alternatives, or by showing that four of them are not correct. This can be a sensible thing to do in the context of the SMC.

However, this does not provide a full mathematical explanation that would be acceptable if you were just given the question without any alternative answers. So for each question we have included a complete solution which does not use the fact that one of the given alternatives is correct.Thus we have aimed to give full solutions with all steps explained. We therefore hope that these solutions can be used as a model for the type of written solution that is expected when presenting a complete solution to a mathematical problem (for example, in the British Mathematical Olympiad and similar competitions).

Enquiries about the Senior Mathematical Challenge should be sent to:

UKMT, School of Mathematics Satellite,
University of Leeds, Leeds LS2 9JT

☎ *0113 343 2339* enquiry@ukmt.org.uk www.ukmt.org.uk

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
A	C	D	C	E	E	B	B	A	B	A	B	D	D	B	D	A	C	C	E	B	C	A	E	D

1. Which of these is the largest number?

A $2+0+1+3$ B $2\times 0+1+3$ C $2+0\times 1+3$ D $2+0+1\times 3$ E $2\times 0\times 1\times 3$

Solution

A We calculate the value of each of the given options in turn.

(a) $2+0+1+3=6$

(b) $2\times 0+1+3=0+1+3=4$

(c) $2+0\times 1+3=2+0+3=5$

(d) $2+0+1\times 3=2+0+3=5$

(e) $2\times 0\times 1\times 3=0$

So option A gives the largest number.

Remarks

You may have obtained the wrong answer if you interpreted $2+0+1\times 3$ to mean $((2+0)+1)\times 3$ rather than $2+0+(1\times 3)$. It is a standard convention (sometimes known as BIDMAS or BODMAS) that in evaluating an expression such as $2+0+1\times 3$, the multiplications are carried out before the additions. We do not just carry out the operations from left to right. So in calculating $2+0+1\times 3$, the multiplication 1×3 is done before the additions.

A decent calculator will produce the correct answer 5 if you press the keys

[2] [+] [0] [+] [1] [×] [3] [=]

in this order. If your calculator produces a different answer, you should replace it!

2. Little John claims he is 2 m 8 cm and 3 mm tall.

What is this height in metres?

A 2.83 m B 2.803 m C 2.083 m D 2.0803 m E 2.0083 m

Solution

C One metre is 100 centimetres. So 1 cm = 0.01 m and 8 cm = 0.08 m. Similarly, one metre is 1000 millimetres. So 1 mm = 0.001 m and 3 mm = 0.003 m. Therefore Little John's height is 2 m + 0.08 m + 0.003 m = 2.083 m.

3. What is the 'tens' digit of $2013^2 - 2013$?

A 0 B 1 C 4 D 5 E 6

Solution

D The 'tens' digit of $2013^2 - 2013$ is the same as that of $13^2 - 13$. Since $13^2 - 13 = 169 - 13 = 156$, the 'tens' digit of $2013^2 - 2013$ is 5.

Remarks

Our comment that the 'tens' digit of $2013^2 - 2013$ is the same as that of $13^2 - 13$ uses the fact that $2013^2 - 2013 = (2000 + 13)^2 - 2013 = (2000^2 + 2 \times 2000 \times 13 + 13^2) - (2000 + 13) = 4000000 + 52000 - 2000 + 13^2 - 13$. It is now clear that only the last two terms, that is, $13^2 - 13$, can have any effect on the 'tens' and 'units' digits of the answer.

A more sophisticated way to say this is to use the language and notation of *modular arithmetic*, which you may already have met. Using this notation we write $a \equiv b \pmod{n}$ to mean that a and b have the same remainder when divided by n. For example, $2013 \equiv 13 \pmod{100}$ and $156 \equiv 56 \pmod{100}$. Then we can say that $2013^2 - 2013 \equiv 13^2 - 13 \pmod{100}$ and $13^2 - 13 \equiv 56 \pmod{100}$. It follows that $2013^2 - 2013 \equiv 56 \pmod{100}$. Thus $2013^2 - 2013$ has remainder 56 when divided by 100. So its last two digits are 5 and 6. In particular its 'tens' digit is 5 and its 'units' digit is 6.

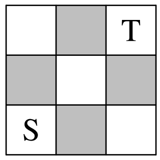
For investigation

3.1 Find the 'tens' digits of (a) $2014^2 - 2014$ and (b) $2013^3 - 2013^2$.

3.2 Find the 'tens' and 'units' digits of (a) 2011^{2011} and (b) 2013^{2013}.

4. A route on the 3×3 board shown consists of a number of steps. Each step is from one square to an adjacent square of a different colour.

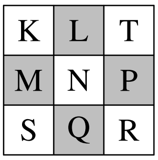
How many different routes are there from square S to square T which pass through every other square exactly once?

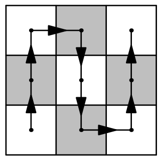


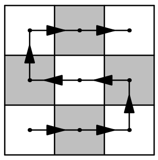
A 0 B 1 C 2 D 3 E 4

SOLUTION

C For convenience, label the other squares as in the left-hand figure. The first move of a route from S to T must be either S→M or S→Q. It is easy to see that a route that visits all the squares must include both the sequence M→K→L and the sequence Q→R→P. Hence, we see that there are just two routes that meet all the required conditions; these are shown in two figures on the the right.

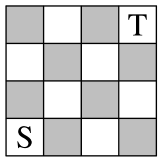






FOR INVESTIGATION

4.1 Consider the analogous problem for a 4×4 board. How many different routes are there from square S to square T which pass through every other square exactly once?



4.2 Now consider the analogous problem for a 5×5 board.

4.3 Now consider the general case of an $n \times n$ board.

NOTE

By considering the problem with a 4×4 board, you should see that the case where n is even is not difficult. However, the case where n is odd is seemingly much more difficult, and we don't know a general formula for this case. Please let us know if you manage to make any progress with this.

5. The numbers x and y satisfy the equations $x(y + 2) = 100$ and $y(x + 2) = 60$.

What is the value of $x - y$?

A 60 B 50 C 40 D 30 E 20

SOLUTION

E The two equations expand to give $xy + 2x = 100$ and $xy + 2y = 60$. It follows that $(xy + 2x) - (xy + 2y) = 100 - 60$. That is, $2x - 2y = 40$. Hence $2(x - y) = 40$ and so $x - y = 20$.

FOR INVESTIGATION

5.1 The wording of the question *implies* that there are numbers x and y which satisfy the equations $x(y+2) = 100$ and $y(x+2) = 60$. Check that this is correct by finding real numbers x and y which satisfy both the equations $x(y+2) = 100$ and $y(x+2) = 60$.

5.2 Show that there are no real numbers x and y such that $x(y+2) = -100$ and $y(x+2) = -80$.

NOTE

There are *complex* number solutions of these equations, and, if x and y are complex numbers which satisfy these equations, then $x - y = -10$. [If you don't know what complex numbers are, ask your teacher.]

6. Rebecca went swimming yesterday. After a while she had covered one fifth of her intended distance. After swimming six more lengths of the pool, she had covered one quarter of her intended distance.

How many lengths of the pool did she intend to complete?

A 40 B 72 C 80 D 100 E 120

SOLUTION

E We have $\frac{1}{4} - \frac{1}{5} = \frac{1}{20}$. So the six additional lengths make up $\frac{1}{20}$ of Rebecca's intended distance. So the number of lengths she intended to complete was $20 \times 6 = 120$.

7. In a 'ninety nine' shop all items cost a number of pounds and 99 pence. Susanna spent £65.76.

How many items did she buy?

A 23 B 24 C 65 D 66 E 76

SOLUTION

B Let k be the number of items that Susanna bought. The cost of these is a whole number of pounds and $99k$ pence, that is, a whole number of pounds less k pence. Susanna spent £65.76, that is, a whole number of pounds less 24 pence. It follows that k pence is a whole number of pounds plus 24 pence. So k is 24 or 124 or 224 or . . . . . However, since each item costs at least 99 pence and Susanna spent £65.76 pence, she bought at most 66 items. So k is 24.

FOR INVESTIGATION

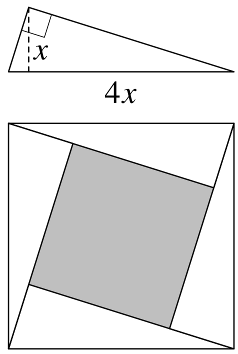
7.1 Is it possible to spend £20.76 in a 'ninety nine' shop?

7.2 For which non-negative integers m and n with $n < 100$ is it possible to spend m pounds and n pence in a 'ninety-nine' shop?

8. The right-angled triangle shown has a base which is 4 times its height. Four such triangles are placed so that their hypotenuses form the boundary of a large square as shown.

What is the side length of the shaded square in the diagram?

A $2x$ B $2\sqrt{2}x$ C $3x$ D $2\sqrt{3}x$ E $\sqrt{15}x$



Solution

B The side length of the large square is $4x$ and hence the area of this square is $16x^2$. Each triangle has base $4x$ and height x and hence has area $\frac{1}{2}(4x \times x) = 2x^2$. So the total area of these four triangles is $8x^2$. Therefore the area of the shaded square is $16x^2 - 8x^2 = 8x^2$. Therefore the side length of the shaded square is $\sqrt{8x^2} = \sqrt{8}x = 2\sqrt{2}x$.

9. According to a headline 'Glaciers in the French Alps have lost a quarter of their area in the past 40 years'.

What is the approximate percentage reduction in the length of the side of a square when it loses one quarter of its area, thereby becoming a smaller square?

A 13% B 25% C 38% D 50% E 65%

Solution

A Suppose that a square of side length 1, and hence area 1, has side length x when it loses one quarter of its area. Then $x^2 = \frac{3}{4}$ and so $x = \frac{\sqrt{3}}{2}$. Now $1.7^2 = 2.89$ and so $1.7 < \sqrt{3}$. Hence $0.85 < \frac{\sqrt{3}}{2}$. So the length of the side of the smaller square is at least 85% of its original value. Therefore the reduction in its length is less than 15%. So, of the given options, it must be that 13% is correct.

For investigation

9.1 The above solution is good enough in the context of the SMC. However, if you couldn't assume that one of the options must be correct, then, to show that the approximate percentage reduction is 13%, you would need to show that the percentage reduction lies between 12.5% and 13.5%. That is, you would need to show that $0.865 < \frac{\sqrt{3}}{2} < 0.875$, or, equivalently, that $1.73 < \sqrt{3} < 1.75$.

You could do this by doing a calculation to show that $1.73^2 < 3 < 1.75^2$. Alternatively, you could calculate $\sqrt{3}$ to the appropriate number of decimal places. This is easy with a calculator, but what could you do if you don't have a calculator? There is an old fashioned method for calculating square roots by hand which resembles long division. This was taught in schools until about fifty years ago. See if you can find out what this method is, perhaps using the internet.

Then use this method to calculate $\sqrt{3}$ to three decimal places.

9.2 The sequence defined by

$$x_1 = 1, \quad x_{n+1} = \frac{x_n + 3}{x_n + 1}$$

gives better and better approximations to $\sqrt{3}$. (We say that it *converges* to $\sqrt{3}$). The first three terms are $x_1 = 1$, $x_2 = \frac{1+3}{1+1} = 2$, and $x_3 = \frac{2+3}{2+1} = \frac{5}{3}$.

(a) Find the values of x_4 and x_5.

(b) Show that if the sequence converges, then it converges to $\sqrt{3}$.

(c) Find a similar rule for a sequence that converges to $\sqrt{5}$.

9.3 The *Generalized Binomial Theorem*, discovered by Isaac Newton, tells us that if $|x| < 1$, then

$$(1 + x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha - 1)}{2!}x^2 + \frac{\alpha(\alpha - 1)(\alpha - 2)}{3!}x^3 + \cdots$$

Use the first *three* terms of this series with $x = -\frac{1}{4}$ and $\alpha = \frac{1}{2}$ to obtain an approximation to $\frac{\sqrt{3}}{2}$.

9.4 What method does your calculator use to evaluate square roots?

10. Frank's teacher asks him to write down five integers such that the median is one more than the mean, and the mode is one greater than the median. Frank is also told that the median is 10.

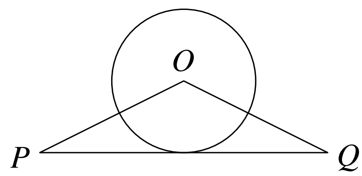
What is the smallest possible integer that he could include in his list?

A 3 B 4 C 5 D 6 E 7

Solution

B The median of the five numbers is 10, hence the mean is 9 and the mode is 11. For the mode to be 11, Frank's list must include more 11s than any other integer. It cannot include three or more 11s, since then the median would be 11. So two of the integers are 11, and the other three integers are all different. For the median to be 10, 10 must be one of the integers, and the list must include two integers that are less than 10 and two that are greater than 10. Hence the five integers are a, b, 10, 11, 11 where a, b are two different integers both less than 10. Say $a < b$. Since the mean is 9, we have $a + b + 10 + 11 + 11 = 5 \times 9 = 45$ and therefore $a + b = 13$. The largest value that b can take is 9, and hence the smallest possible value for a is 4.

11. The diagram shows a circle with centre O and a triangle OPQ. Side PQ is a tangent to the circle. The area of the circle is equal to the area of the triangle.



What is the ratio of the length of PQ to the circumference of the circle?

A $1 : 1$ B $2 : 3$ C $2 : \pi$ D $3 : 2$ E $\pi : 2$

Solution

A Suppose that the circle has radius r and that PQ has length x. The height of the triangle OPQ is the length of the perpendicular from O to PQ. Since PQ is a tangent to the circle, this perpendicular is a radius of the circle and so has length r. Therefore the area of triangle OPQ is $\frac{1}{2}xr$. The area of the circle is πr^2. Since these areas are equal $\frac{1}{2}xr = \pi r^2$, and hence $x = 2\pi r$. So the length of PQ is the same as the circumference of the circle. So the ratio of their lengths is $1 : 1$.

12. As a special treat, Sammy is allowed to eat five sweets from his very large jar which contains many sweets of each of three flavours – Lemon, Orange and Strawberry. He wants to eat his five sweets in such a way that no two consecutive sweets have the same flavour.

In how many ways can he do this?

A 32 B 48 C 72 D 108 E 162

Solution

B Sammy has a choice of 3 flavours for the first sweet that he eats. Each of the other sweets he eats cannot be the same flavour as the sweet he has just eaten. So he has a choice of 2 flavours for each of these four sweets. So the total number of ways that he can make his choices is $3 \times 2 \times 2 \times 2 \times 2 = 48$.

For investigation

12.1 What would the answer be if Sammy had sweets with four different flavours in his jar?

12.2 Find a formula for the number of ways if Sammy is allowed to eat k sweets and has sweets of n different flavours in his jar.

13. Two entrants in a school's sponsored run adopt different tactics. Angus walks for half the time and runs for the other half, whilst Bruce walks for half the distance and runs for the other half. Both competitors walk at 3 mph and run at 6 mph. Angus takes 40 minutes to complete the course.

How many minutes does Bruce take?

A 30 B 35 C 40 D 45 E 50

Solution

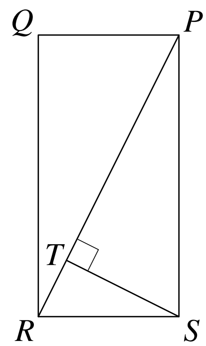
D Angus walks for 20 minutes at 3 mph and runs for 20 minutes at 6 mph. 20 minutes is one-third of an hour. So the number of miles that Angus covers is $3 \times \frac{1}{3} + 6 \times \frac{1}{3} = 3$.

Bruce covers the same distance. So Bruce walks $\frac{1}{2} \times 3$ miles at 3 mph which takes him 30 minutes and runs the same distance at 6 mph which takes him 15 minutes. So altogether it takes Bruce 45 minutes to finish the course.

14. The diagram shows a rectangle $PQRS$ in which $PQ : QR = 1 : 2$. The point T on PR is such that ST is perpendicular to PR.

What is the ratio of the area of the triangle RST to the area of the rectangle $PQRS$?

A $1 : 4\sqrt{2}$ B $1 : 6$ C $1 : 8$ D $1 : 10$ E $1 : 12$



Solution

D We can suppose that we have chosen units so that the length of RS is 1. So the length of PS is 2. It follows from Pythagoras' theorem that the length of PR is $\sqrt{1^2 + 2^2}$, that is, $\sqrt{5}$.

The triangles RST and PRS are both right-angled and the angle at R is common to both triangles. Therefore the triangles are similar. Hence the ratio of their areas is the ratio of the squares of the lengths of corresponding sides. The lengths of their hypotenuses are in the ratio $1 : \sqrt{5}$. Hence, the ratio of the area of triangle RST to the area of triangle PRS is $1^2 : \sqrt{5}^2$, that is, $1 : 5$. The area of triangle PRS is half the area of the rectangle. Hence the ratio of the area of triangle RST to the area of the rectangle is $1 : 10$.

For investigation

14.1 The solution shows that the ratio of the area of triangle RST to the area of the rectangle $PQRS$ is $1 : 10$. Is it possible to prove this by dissecting the rectangle $PQRS$ into 10 triangles each congruent to the triangle RST?

15. For how many positive integers n is $4^n - 1$ a prime number?

A 0 B 1 C 2 D 3 E infinitely many

Solution

B We give two methods for answering this question. Both methods rely on the following property of prime numbers: a positive integer p is a prime number if, and only if, the only way p can be expressed as a product of two positive integers, m and n, is where exactly one of m and n is 1.

Method 1

The first method uses the fact that when n is a positive integer $4^n - 1$ is the difference of two squares and so may be factorized as $4^n - 1 = (2^n)^2 - 1^2 = (2^n + 1)(2^n - 1)$. So $4^n - 1$ is not a prime, unless one of the factors $2^n + 1$ and $2^n - 1$ is equal to 1. Now $2^n + 1$ cannot be equal to 1 and $2^n - 1 = 1$ if, and only if, $n = 1$. When $n = 1$, we have $4^n - 1 = 3$ and so $4^n - 1$ is prime. So there is just one positive integer n, namely 1, for which $4^n - 1$ is a prime number.

Method 2

The second method uses the fact when n is a positive integer $x - 1$ is a factor of $x^n - 1$, since, for $n \geq 2$,

$$x^n - 1 = (x - 1)(x^{n-1} + x^{n-2} + \cdots + x + 1).$$

Hence, putting $x = 4$, we deduce that, for each positive integer n, 3 is a factor of $4^n - 1$. Therefore $4^n - 1$ is not prime except when $n = 1$ and $4^n - 1 = 3$. So there is just one positive integer n for which $4^n - 1$ is a prime number.

For investigation

15.1 For how many positive integers n is $5^n - 1$ a prime number?

15.2 For how many positive integers n is $6^n - 1$ a prime number?

15.3 Prove the identity used in method 2, that is, that for all integers $n \geq 2$,

$$x^n - 1 = (x - 1)(x^{n-1} + x^{n-2} + \cdots + x + 1).$$

16. Andrew states that every composite number of the form $8n + 3$, where n is an integer, has a prime factor of the same form.

Which of these numbers is an example showing that Andrew's statement is false?

A 19 B 33 C 85 D 91 E 99

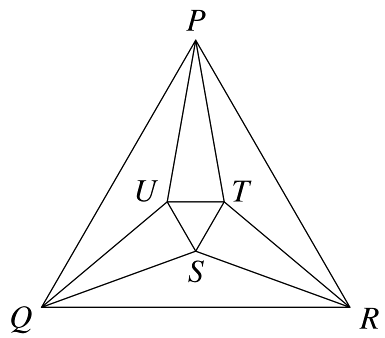
SOLUTION

D To show that Andrew's statement is wrong we need to find a composite number of the form $8n + 3$ which does not have any prime factors of this form. This rules out 19 which is not composite, and both 33 and 85 which are not of the form $8n + 3$. The number 99 is of the form $8n + 3$, but it has two prime factors 3 and 11 which are also of this form. So 99 won't do either. This leaves 91. We see that $91 = 8 \times 11 + 3$, and so it is of the form $8n + 3$, but $91 = 7 \times 13$ and neither of its prime factors, 7 and 13, is of this form. So the number 91 shows that Andrew's statement is false.

17. The equilateral triangle PQR has side length 1. The lines PT and PU trisect the angle RPQ, the lines RS and RT trisect the angle QRP and the lines QS and QU trisect the angle PQR.

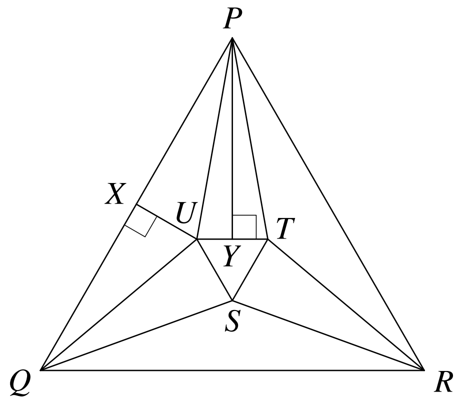
What is the side length of the equilateral triangle STU?

A $\dfrac{\cos 80^\circ}{\cos 20^\circ}$ B $\frac{1}{3}\cos 20^\circ$ C $\cos^2 20^\circ$ D $\frac{1}{6}$ E $\cos 20^\circ \cos 80^\circ$



Solution

A Since the triangle PQR is equilateral, all its three angles are 60°. So the trisectors divide these into three 20° angles. In particular $\angle QPU = 20^\circ$ and $\angle UPT = 20^\circ$. Because of the symmetry of the figure, $QU = PU = PT$. So the triangles PQU and PUT are isosceles. In these triangles, we let X be the foot of the perpendicular from U to PQ, and Y be the foot of perpendicular from P to UT.



In the right-angled triangle PXU, $\angle XPU = 20^\circ$, and so $\frac{PX}{PU} = \cos 20^\circ$. Since the length of PX is $\frac{1}{2}$ it follows that PU has length $\frac{1}{2} \div \cos 20^\circ$. In the right-angled triangle PYU, $\angle UPY = 10^\circ$ and therefore $\angle YUP = 80^\circ$.

Therefore, from triangle PYU, $\cos 80^\circ = \frac{UY}{PU}$ and hence UY has length $PU \cos 80^\circ$, that is,

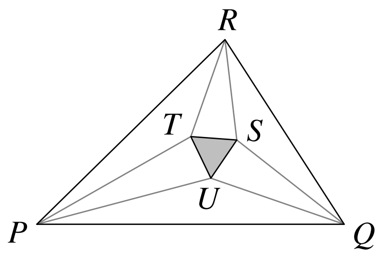
$$\frac{1}{2} \times \frac{\cos 80^\circ}{\cos 20^\circ}.$$

Since the length of UY is half the length of UT, it follows that the length of UT is $\dfrac{\cos 80^\circ}{\cos 20^\circ}$.

Remarks

In this problem, it is clear that, because the triangle PQR is equilateral, so also is the triangle STU. It is a remarkable fact, discovered by Frank Morley (1860–1937), that:

In every triangle PQR, the triangle STU formed by the points where the trisectors of the angles of PQR meet is an equilateral triangle.



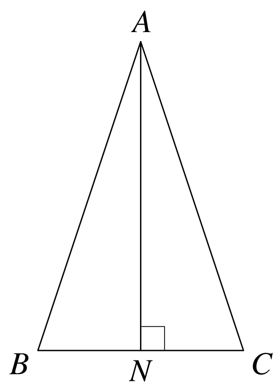
The triangle STU is called the *Morley triangle* of the triangle PQR.

No very easy proofs of this theorem are known.

A proof may be found on pages 186–187 of *The Geometry of the Triangle* by Gerry Leversha, UKMT, 2013. This book may be ordered from the UKMT website. The virtue of the proof in this book is that it uses purely geometrical ideas. Other proofs use trigonometry and complex numbers.

For investigation

17.1 In the solution above we have assumed that X and Y are the midpoints of PQ and UT, respectively, and that $\angle UPY = \angle TPY$. These facts both follow from the fact that if a triangle ABC is isosceles with $AB = AC$, and N is the foot of the perpendicular from A to BC, then the triangles ABN and ACN are congruent.



Can you prove this?

Note that it then follows that $BN = CN$ and $\angle BAN = \angle CAN$. So we can deduce that X, Y are the midpoints of PQ and UT, and that $\angle UPY = \angle TPY$, as required.

17.2 An alternative method is to apply the Sine Rule to the triangles PQU and PUT. This gives

$$\frac{PU}{\sin 20°} = \frac{PQ}{\sin 140°} \quad \text{and} \quad \frac{UT}{\sin 20°} = \frac{PU}{\sin 80°}.$$

Hence, since PQ has length 1, it follows that

$$UT = PU \times \frac{\sin 20°}{\sin 80°} = \frac{\sin 20°}{\sin 140°} \times \frac{\sin 20°}{\sin 80°} = \frac{\sin^2 20°}{\sin 140° \sin 80°}.$$

Show, *without using a calculator*, that this is the same as the previous answer $\dfrac{\cos 80°}{\cos 20°}$.

18. The numbers 2, 3, 12, 14, 15, 20, 21 may be divided into two sets so that the product of the numbers in each set is the same.

What is this product?

A 420 B 1260 C 2520 D 6720 E 6 350 400

SOLUTION

C If all the numbers 2, 3, 12,14, 15, 20, 21 are multiplied the result will be the square of the common product of the two sets with the same product. Now

$$\begin{aligned}2 \times 3 \times 12 \times 14 \times 15 \times 20 \times 21 &\\ &= 2 \times 3 \times (2^2 \times 3) \times (2 \times 7) \times (3 \times 5) \times (2^2 \times 5) \times 3 \times 7\\ &= 2^6 \times 3^4 \times 5^2 \times 7^2\\ &= (2^3 \times 3^2 \times 5 \times 7)^2.\end{aligned}$$

Therefore the common product is $2^3 \times 3^2 \times 5 \times 7 = 2520$.

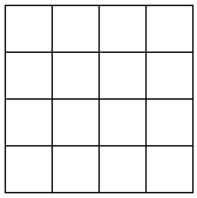
FOR INVESTIGATION

18.1 This solution only shows that *if the common product exists* then it equals 2520. A complete solution should also show that it is possible to split the numbers 2, 3, 12, 14, 15, 20, 21 into two sets, each of whose products in 2520. Show that this is possible, and in just one way.

18.2 Devise some other problems of the same type as this one.

19. The 16 small squares shown in the diagram each have a side length of 1 unit.

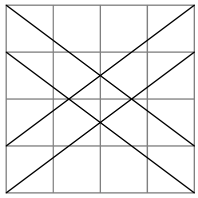
How many pairs of vertices are there in the diagram whose distance apart is an integer number of units?

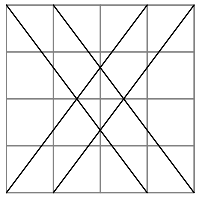


A 40 B 64 C 108 D 132 E 16

SOLUTION

C Each pair of distinct vertices taken from the same row or from the same column is an integer number of units (1, 2, 3 or 4 units) apart. There are 5 vertices in each row and each column, and so 2 vertices may be chosen from the same row or the same column in $\binom{5}{2} = 10$ ways. Since there are 5 rows and 5 columns this gives $50 + 50 = 100$ pairs of vertices that are an integer number of units apart.





In addition, since $3^2 + 4^2 = 5^2$, it follows from Pythagoras' theorem that the opposite vertices of a 3×4 rectangle are 5 units apart. This gives a further 8 pairs of vertices (forming the end points of the 8 diagonals shown in the diagrams) which are an integer number of units apart.

There are no other *Pythagorean triples*, that is, positive integers m, n, q such that $m^2 + n^2 = q^2$, where m and n are both less than 5. So we have found all the pairs of vertices that are an integer number of units apart. So there are altogether $100 + 8 = 108$ pairs of such vertices.

20. The ratio of two positive numbers equals the ratio of their sum to their difference.

What is this ratio?

A $(1 + \sqrt{3}) : 2$ B $\sqrt{2} : 1$ C $(1 + \sqrt{5}) : 2$ D $(2 + \sqrt{2}) : 1$ E $(1 + \sqrt{2}) : 1$

SOLUTION

E Let the two numbers be x and y and suppose that $x : y = k : 1$, so that $x = ky$. Since the ratio $x + y : x - y$ is also $k : 1$, it follows that $x + y = k(x - y)$ and so $ky + y = k(ky - y)$. Since y is positive, we may divide through by y to obtain $k + 1 = k(k - 1)$.

It follows that $k^2 - 2k - 1 = 0$, and so, using the standard formula for the roots of a quadratic equation, we get

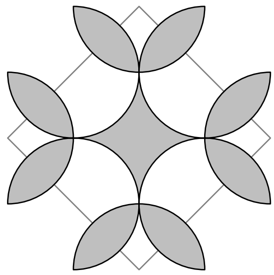
$$k = \frac{2 \pm \sqrt{4 + 4}}{2} = 1 \pm \sqrt{2}.$$

Since x and y are both positive $k > 0$, and it follows that $k = 1 + \sqrt{2}$.

21. The shaded design shown in the diagram is made by drawing eight circular arcs, all with the same radius. The centres of four arcs are the vertices of the square; the centres of the four touching arcs are the midpoints of the sides of the square. The diagonals of the square have length 1.

What is the total length of the border of the shaded design?

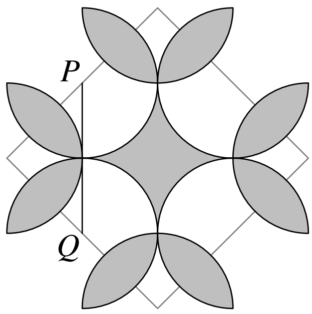
A 2π B $\frac{5}{2}\pi$ C 3π D $\frac{7}{2}\pi$ E 4π



Solution

B The arcs whose centres are the vertices of the square are semi-circular arcs, and the arcs whose centres are the midpoints of the sides of the square form three-quarter circles. Let all these arcs have radius r. Then the length of the border is

$$4 \times \left(\tfrac{1}{2} \times 2\pi r\right) + 4 \times \left(\tfrac{3}{4} \times 2\pi r\right) = 10\pi r.$$



Consider the line PQ joining the midpoints of two adjacent sides of the square as shown. Clearly the length of PQ is half of the length of the diagonal of the square. So PQ has length $\frac{1}{2}$. PQ joins the midpoints of two touching arcs and so its length is twice the radius of these arcs. Hence $r = \frac{1}{4}$. Therefore the length of the border of the shaded design is $10\pi \times \frac{1}{4} = \frac{5}{2}\pi$.

For investigation

21.1 The above solution assumes that the arcs whose centres are the vertices of the square are semi-circles, and the arcs whose centres are the midpoints of the sides of the square form three-quarter circles. Give arguments to prove that these assumptions are correct.

22. Consider numbers of the form $10n + 1$, where n is a positive integer. We shall call such a number 'grime' if it cannot be expressed as the product of two smaller numbers, possibly equal, both of which are of the form $10k + 1$, where k is a positive integer.

How many 'grime numbers' are there in the sequence 11, 21, 31, 41, ..., 981, 991?

A 0 B 8 C 87 D 92 E 99

Solution

C Instead of checking each of the 99 numbers in the sequence 11, 21, ..., 981, 991 in turn to see whether or not they are 'grime numbers', it is easier to count the numbers in this sequence that are not 'grime numbers'. A number is not a 'grime number' if, and only if, it is a product, say rs, where both r and s are numbers of the form $10k + 1$, where k is a positive integer. We can assume that $r \leq s$, and since we are only interested in numbers rs such that $rs \leq 991$, we can also assume that $r \leq 31$, because $32^2 > 991$.

We see that the only products of this form which are not greater than 991 are the twelve numbers

$$11 \times 11, \quad 11 \times 21, \quad 11 \times 31, \quad 11 \times 41, \quad 11 \times 51, \quad 11 \times 61,$$
$$11 \times 71, \quad 11 \times 81, \quad 21 \times 21, \quad 21 \times 31, \quad 21 \times 41 \quad \text{and} \quad 31 \times 31.$$

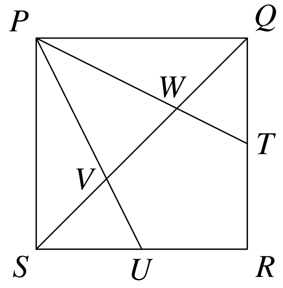
It can be seen, without calculating their values, that all these products are different as they have different prime factorisations.

So there are 12 numbers in the sequence 11, 21, ..., 981, 991 that are not 'grime numbers'. Hence there are $99 - 12 = 87$ 'grime numbers' in this sequence.

23. $PQRS$ is a square. The points T and U are the midpoints of QR and RS respectively. The line QS cuts PT and PU at W and V respectively.

What fraction of the total area of the square $PQRS$ is the area of the pentagon $RTWVU$?

A $\frac{1}{3}$ B $\frac{2}{5}$ C $\frac{3}{7}$ D $\frac{5}{12}$ E $\frac{4}{15}$



SOLUTION

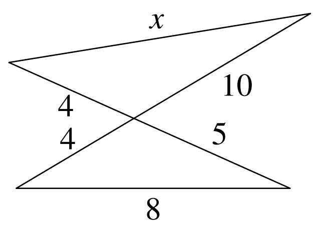
A The lines PQ and SR are parallel. Hence $\angle USV = \angle PQV$, since they are alternate angles. Similarly $\angle SUV = \angle PQV$. It follows that the triangles USV and QPV are similar. Now $SU : PQ = 1 : 2$ and so the heights of these triangles are in the same ratio. So the height of triangle USV is $\frac{1}{3}$ of the side-length of the square. The base of this triangle is $\frac{1}{2}$ of the side of the square. Hence the area of this triangle is $\frac{1}{2}\left(\frac{1}{2} \times \frac{1}{3}\right) = \frac{1}{12}$ of the area of the square.

Similarly the area of triangle QWT is $\frac{1}{12}$ of the area of the square. The area of triangle PQS is $\frac{1}{2}$ of the area of the square. The area of the pentagon $RTWVU$ is the area of the square minus the total areas of the triangles USV, QWT and PQS, so its area, as a fraction of the area of the square $PQRS$, is $1 - \frac{1}{12} - \frac{1}{12} - \frac{1}{2} = \frac{1}{3}$.

24. The diagram shows two straight lines PR and QS crossing at O.

What is the value of x?

A $7\sqrt{2}$ B $2\sqrt{29}$ C $14\sqrt{2}$
D $7(1+\sqrt{13})$ E $9\sqrt{2}$



Solution

E Let $\angle SOR = \theta$. Applying the Cosine Rule to the triangle ROS, we obtain

$$8^2 = 4^2 + 5^2 - 2 \times 4 \times 5 \times \cos\theta,$$

from which it follows that

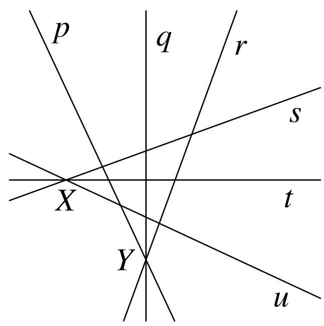
$$\cos\theta = \frac{4^2 + 5^2 - 8^2}{2 \times 4 \times 5} = -\frac{23}{40}.$$

We also have that $\angle QOP = \angle SOR = \theta$, since they are vertically opposite angles. Hence, applying the Cosine Rule to triangle POQ, we get

$$\begin{aligned} x^2 &= 4^2 + 10^2 - 2 \times 4 \times 10 \times \cos\theta \\ &= 16 + 100 - 80\left(-\frac{23}{40}\right) \\ &= 116 + 46 = 162. \end{aligned}$$

Therefore $x = \sqrt{162} = \sqrt{81 \times 2} = 9\sqrt{2}$.

25. Challengeborough's underground train network consists of six lines p, q, r, s, t and u, as shown. Wherever two lines meet there is a station which enables passengers to change lines. On each line, each train stops at every station.



Jessica wants to travel from station X to station Y. She does not want to use any line more than once, nor return to station X after leaving it, nor leave station Y after reaching it.

How many different routes, satisfying these conditions, can she choose?

A 9 B 36 C 41 D 81 E 720

Solution

D A route is specified by giving the sequence of lines that Jessica travels on. We call the lines s, t and u the X-lines and the lines p, q and r the Y-lines. It follows from the layout of the network and Jessica's conditions, that she can change trains between any X-line and any Y-line and *vice versa*, but she cannot change between two X-lines, or between two Y-lines. So her route from X to Y is given by a sequence of lines starting with an X-line, alternating between X-lines and Y-lines, and ending with a Y-line. So it will consist of an even number of lines. Since Jessica does not wish to use any line more than once, a possible route for Jessica consists of 2, 4 or 6 lines. We count these routes according to the number of lines involved.

2 lines A route of 2 lines will be of the form g, h, where g is an X-line and h is a Y-line. There are 3 choices for g and 3 choices for h, and hence $3 \times 3 = 9$ routes of this form.

4 lines A route of 4 lines will be of the form g, h, i, j, where g and i are two different X-lines and h and j are two different Y-lines. Since there are 3 choices for g and then 2 choices for i and likewise for h and j, there are $3 \times 3 \times 2 \times 2 = 36$ routes of this form.

6 lines A route of 6 lines will have the form g, h, i, j, k, l, where g, i and k are X-lines and h, j and l are Y-lines. As before there are 3 choices for g and then 2 choices for i, leaving just one choice for k, and likewise for the choices of h, j and l. So there are $3 \times 3 \times 2 \times 2 \times 1 \times 1 = 36$ routes of this form.

So, altogether, there are $9 + 36 + 36 = 81$ routes that satisfy Jessica's conditions.

For investigation

25.1 How many different routes would there be if there were four lines passing through X and four lines passing through Y, but otherwise the conditions are the same?

25.2 Can you find an expression which gives the number of different routes if there are m lines passing through X and n lines passing through Y, but otherwise the conditions are the same?