UK SENIOR MATHEMATICAL CHALLENGE

November 8th 2011

EXTENDED SOLUTIONS

These solutions augment the printed solutions that we send to schools. For convenience, the solutions sent to schools are confined to two sides of A4 paper and therefore in many cases are rather short. The solutions given here have been extended. In some cases we give alternative solutions, and we have included some ***Extension Problems*** for further investigations.

The Senior Mathematical Challenge (SMC) is a multiple choice contest, in which you are presented with five options, of which just one is correct. It follows that often you can find the correct answers by working backwards from the given alternatives, or by showing that four of them are not correct. This can be a sensible thing to do in the context of the SMC, and we often first give a solution using this approach.

However, this does not provide a full mathematical explanation that would be acceptable if you were just given the question without any alternative answers. So for each question we have included a complete solution which does not use the fact that one of the given alternatives is correct. Thus we have aimed to give full solutions with all steps explained. We therefore hope that these solutions can be used as a model for the type of written solution that is expected when presenting a complete solution to a mathematical problem (for example, in the British Mathematical Olympiad and similar competitions).

We welcome comments on these solutions, and, especially, corrections or suggestions for improving them. Please send your comments,

either by e-mail to: enquiry@ukmt.co.uk

or by post to: SMC Solutions, UKMT Maths Challenges Office, School of Mathematics, University of Leeds, Leeds LS2 9JT.

Quick Marking Guide

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
D	D	A/B	D	B	C	D	C	B	C	C	D	D	C	B	E	A	E	B	B	C	A	B	B	C

supported by

The Actuarial Profession

making financial sense of the future

1. Which of the numbers below is not a whole number?

A $\frac{2011+0}{1}$ B $\frac{2011+1}{2}$ C $\frac{2011+2}{3}$ D $\frac{2011+3}{4}$ E $\frac{2011+4}{5}$

Solution: **D**

It is straightforward to check that $\frac{2011+3}{4}=\frac{2014}{4}=\frac{1007}{2}$ is not a whole number, but that all the other fractions given above are whole numbers. A more general approach to the question is as follows. We have that $\frac{2011+(n-1)}{n}=\frac{2010+n}{n}=\frac{2010}{n}+1$, and therefore, $\frac{2011+(n-1)}{n}$ is a whole number if and only if $\frac{2010}{n}$ is a whole number, that is, if and only if n is a factor of 2010. Since 2010 has the prime factorization $2\times 3\times 5\times 67$, its distinct factors are 1, 2, 3, 5, 6, 10, 15, 30, 67, 134, 201, 335, 402, 670, 1005 and 2010. These are the only positive whole number values of n for which $\frac{2011+(n-1)}{n}$ is also a whole number.

Extension Problems

1.1 List the positive whole numbers for which $\frac{2013+(n-1)}{n}$ is an integer.

1.2 Find all the positive whole numbers, k, such that for $n=1$, 2, 3 and 5, but *not* for $n=4$, $\frac{k+(n-1)}{n}$ is an integer.

1.3 Let r be a positive whole number**.** Find all the positive whole numbers, k, such that for all positive integers $n\le r$, $\frac{k+(n-1)}{n}$ is a whole number.

2. Jack and Jill went up the hill to fetch a pail of water. Having filled the pail to the full, Jack fell down spilling $\frac{2}{3}$ of the water, before Jill caught the pail. She then tumbled down the hill, spilling $\frac{2}{5}$ of the remainder.

What fraction of the pail does the remaining water fill?

A $\frac{11}{15}$ B $\frac{1}{3}$ C $\frac{4}{15}$ D $\frac{1}{5}$ E $\frac{1}{15}$

Solution: **D**

After Jack spills $\frac{2}{3}$ of the water, there remains $\frac{1}{3}$. Jill spills $\frac{2}{5}$ of this, leaving $\frac{3}{5}$ of it. So there remains $\frac{3}{5}\times\frac{1}{3}=\frac{1}{5}$ of the water.

3. The robot *Lumber9* moves along the number line. *Lumber9* starts at 0, takes 1 step forward (to 1), then 2 steps backwards (to -1), then 3 steps forward, 4 steps back, and so on, moving alternately forwards and backwards, one more step each time. At what number is *Lumber9* after 2011 steps?

A 1006 B 27 C 11 D 0 E -18

Note: Unfortunately, the word "step" in the final sentence of this question might be interpreted in two different ways. We apologize for this ambiguity.

On one reading, the question asks at what number *Lumber9* is after 2011 *stages* in the process. Solution 1 corresponds to this interpretation.

The question could also mean "At what number is *Lumber9* after it has taken 2011 steps in total." Solution 2 corresponds to this interpretation.

Both solutions were awarded the marks when the paper was marked.

Solution 1: **A**

After 2011 stages *Lumber9* reaches the number $1-2+3-4+\ldots+2009-2010+2011$

$=(1-2)+(3-4)+\ldots+(2009-2010)+2011=(-1-1-\ldots-1)+2011$. The number of -1 s in this sum is $\frac{1}{2}(2010)\ =1005$. So $1-2+3-4+\ldots+2011=1005\times(-1)+2011=2011-1005=1006$.

Solution 2: **B**

After 62 stages of the process *Lumber9* has taken $1+2+3+\ldots+62=1953$ steps and has reached the number $1-2+3-4+\ldots+61-62=-31$. After taking another 58 steps in the positive direction, it has taken $1953+58=2011$ steps in total, and has reached the number $-31+58=27$.

4. What is the last digit of 3^{2011}?

A 1 B 3 C 5 D 7 E 9

Solution: **D**

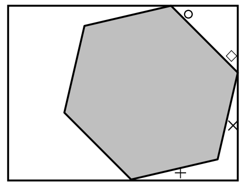
Since $3^4=81$, the last digit of $3^{2008}=(3^4)^{502}$ is 1. Since $3^{2011}=3^{2008+3}=3^{2008}\times 3^3$ $=3^{2008}\times 27$, the last digit of 3^{2011} is the same as the last digit of 27, that is, 7.

[The last digit of a number is its remainder when we divide by 10. So here we are really working with congruences (mod 10). In this language we can write the above calculation as

$3^{2011}\equiv 3^{4\times 502+3}\equiv(3^4)^{502}\times 3^3\equiv 1^{502}\times 7\equiv 7$ (mod 10).]

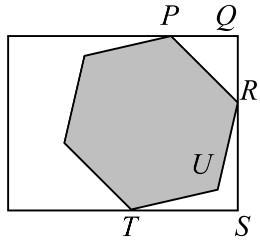
5. The diagram shows a regular hexagon inside a rectangle.

What is the sum of the four marked angles?



A 90^0 B 120^0 C 150^0 D 180^0 E 210^0

Solution: **B**



We label the vertices *P*, *Q*, *R*, *S*, *T* and *U* as shown. In the triangle *PQR* the sum of all three angles is 180^0 and $\angle PQR = 90^0$. Therefore, $\angle QPR + \angle PRQ = 90^0$. In the quadrilateral *RSTU*, the sum of all four angles is 360^0, $\angle RST = 90^0$, and the exterior angle at *U* is 240^0, because the interior angle of a regular hexagon is 120^0. Therefore, $\angle SRU + \angle STU = 360^0 - 90^0 - 240^0 = 30^0$. Therefore the sum of the four marked angles, that is, $\angle QPR + \angle PRQ + \angle SRU + \angle STU$, is $90^0 + 30^0 = 120^0$.

6. Granny and her granddaughter Gill both had their birthday yesterday. Today, Granny's age in years is an even number and 15 times that of Gill. In 4 year's time Granny's age in years will be the square of Gill's age in years. How many years older than Gill is Granny today?

A 42 B 49 C 56 D 60 E 64

Solution: **C**

Suppose that today Gill is *x* years old. It follows that Granny is 15*x* years old. In 4 years time their ages in years will be $x + 4$ and $15x + 4$ respectively. So $15x + 4 = (x+4)^2$. Thus $15x + 4 = x^2 + 8x + 16$. Hence $x^2 - 7x + 12 = 0$, that is, $(x-3)(x-4) = 0$ and so *x* is either 3 or 4. So today Gill is either 3 or 4 years old and Granny is either 45 or 60. Since Granny's age in years is an even number, we deduce that today Granny is 60 and Gill is 4. So the difference in their ages is 56 years.

7. Two sides of a triangle have lengths 4cm and 5cm. The third side has length *x* cm, where *x* is a positive integer. How many different values can *x* have?

A 4 B 5 C 6 D 7 E 8

Solution: **D**

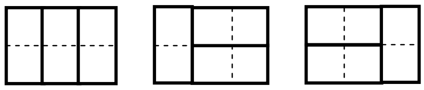
We use two key facts. First, the sum of the lengths of the two shorter sides of a triangle must be greater than the length of the longest side. Second, if *a*, *b, c* are positive numbers with $a \le b \le c$ and $a + b > c$, then there is a triangle whose side lengths are *a*, *b* and *c*.

So if $x \le 4$, we must have that $x + 4 > 5$, so $x > 1$, and so there are 3 possible integer values of x, namely 2, 3 and 4. If $x \ge 5$, we must have that $x < 4 + 5 = 9$, and there are 4 possible integer values of x, namely 5, 6, 7 and 8. So altogether there are 7 possible values for x.

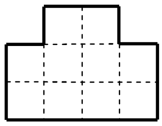
Extension Problem

7.1 If two sides of a triangle have lengths a cm and b *cm*, where a and b are integers with $a \le b$, how many possible integer values are there for the third side of the triangle?

8. A 2×3 grid of squares can be divided into 1×2 rectangles in three different ways.



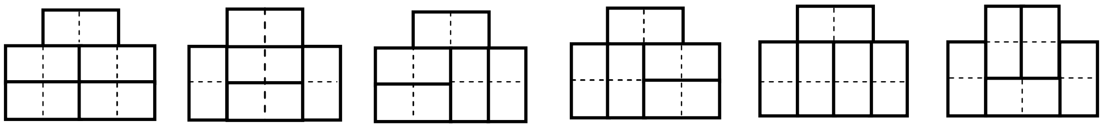
How many ways are there of dividing the bottom shape into 1×2 rectangles?



A 1 B 4 C 6 D 7 E 8

Solution: **C**

In the context of the SMC it is most straightforward just to list all the 6 different layouts:



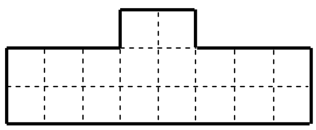
For a deeper analysis of the problem see the Extension Problems below.

Extension Problems

8.1 First consider the number of ways of dividing a $2 \times n$ grid into 1×2 rectangles. Let this number be x_n. The example given in the question shows that $x_3 = 3$. Determine the values of x_n for $n = 1$, 2, 4 and 5. Use these values to make a conjecture about the infinite sequence of values $x_1, x_2, x_3, \ldots$.

8.2 Prove that your conjecture in 8.1 is correct. Hence obtain a formula for x_n.

8.3 Next we consider the number of ways of dividing the type of grid that Question 8 deals with, that is a $2 \times n$ grid, where n is even, augmented with a 1×2 grid on top in the middle. Thus the grid in Question 8 is the case $n = 4$, and the grid below is the case $n = 8$.



In the case where n is even, we let y_n be the number of ways of dividing a $2 \times n$ grid with a 1×2 grid on top in the middle into 1×2 rectangles. By considering first the 1×2 grid on top, show that, when n is even $y_n = x_n + (x_{\frac{1}{2}n-1})^2$.

Use this formula to verify that $y_4 = 6$, and to calculate the values of y_6 and y_8.

8.4 This problem can be generalized to other grids, divided into other rectangles. You are invited to consider some of these generalizations. Most of them are rather intractable. It is suggested that you begin by considering the number of ways of dividing a $3 \times n$ grid into 1×3 rectangles.

9.	Sam has a large collection of $1 \times 1 \times 1$ cubes, each of which is either red or yellow. Sam makes a $3 \times 3 \times 3$ block from twenty-seven cubes, so that no cubes of the same colour meet face-to-face. What is the difference between the largest number of red cubes that Sam can use and the smallest number?
	A 0 B 1 C 2 D 3 E 4

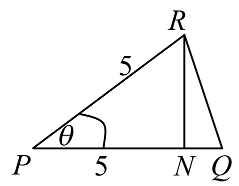
Solution: **B**

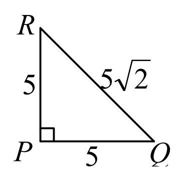
There are only two arrangements that meet the requirement that cubes of the same colour do not meet face to face. Either the centre small cube is red, in which case all the small cubes at the centres of the 12 edges of the large cube are red, with all the other small cubes being yellow, or the colours are the other way round. In the first case there are 13 red cubes and 14 yellow cubes. In the second case there are 14 red cubes and 13 yellow cubes. $14 - 13 = 1$.

10.	A triangle has two edges of length 5. What length should be chosen for the third edge of the triangle so as to maximize the area within the triangle?
	A 5 B 6 C $5\sqrt{2}$ D 8 E $5\sqrt{3}$

Solution: **C**

Let the triangle be *PQR*, where $PQ = PR = 5$, and let *RN* be the perpendicular from *R* to *PQ*. [The diagram illustrates the case where $\angle QPR$ is acute. You should satisfy yourself that the argument is the same in the case where this angle is obtuse.] The area of the triangle is $\frac{1}{2} PQ.RN = \frac{5}{2} RN$ and hence is a maximum when the length of *RN* is a maximum. From the right-angled triangle *PNR* we see that if the points the points *N* and *P* are distinct, $RN < PR$. Hence the maximum possible length of RN is 5 when $\angle RPQ = 90^0$, so that N and *P* coincide.. In this case *RQ* is the hypotenuse of a right-angled isosceles triangle in which the other two sides have length 5, and so, by Pythagoras' Theorem $RQ = 5\sqrt{2}$. Therefore, this is the length of the third side which makes the area a maximum.



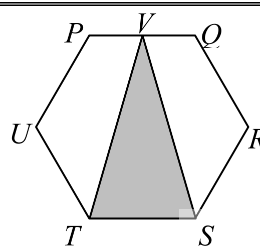


[Another way to obtain this answer is to use trigonometry. If $\angle RPQ = \theta$, the area of the triangle is $\frac{1}{2} 5^2 \sin\theta$. This is a maximum when $\sin\theta = 1$, that is, when $\theta = 90^0$. For a third method see Extension Problem 10.1.]

Extension Problem

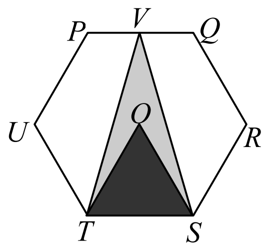
10.1 Show that Question 10 may also be solved using calculus, as follows. Let x be the length of the third side. Use Pythagoras' Theorem to calculate the area of the triangle in terms of x, and then calculus to find the value of x which gives the maximum value of the area.

11. *PQRSTU* is a regular hexagon, and V is the midpoint of PQ. What fraction of the area of *PQRSTU* is the area of triangle *STV*?



A $\frac{1}{4}$ B $\frac{2}{15}$ C $\frac{1}{3}$ D $\frac{2}{5}$ E $\frac{5}{12}$

Solution: **C**



Let O be the centre of the hexagon. The triangle *STO* has the same base as triangle *STV* and half its height. So the area of triangle *STV* is twice that of triangle *STO*. The area of triangle *STO* is $\frac{1}{6}$ of the area of the hexagon. Hence, the area of triangle *STV* is $\frac{1}{3}$ of the area of the hexagon.

12. The *primorial* of a number is the product of all the prime numbers less than or equal to that number. For example, the primorial of 6 is $2 \times 3 \times 5 = 30$. How many different whole numbers have a primorial of 210?

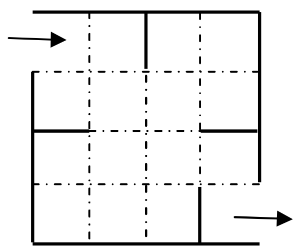
A 1 B 2 C 3 D 4 E 5

Solution: **D**

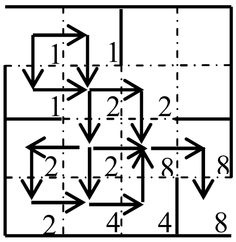
Since $210 = 2 \times 3 \times 5 \times 7$, the primorial of a positive integer n is 210 if and only if the primes less than or equal to n are 2, 3, 5 and 7. This holds if and only if $7 \leq n < 11$, as 11 is the next largest prime after 7. So the primorial of n is 210 just for the four values $n = 7$, 8, 9 and 10.

13. The diagram represents a maze. Given that you can only move horizontally and vertically and are not allowed to revisit a square, how many different routes are there through the maze?

A 16 B 12 C 10 D 8 E 6



Solution: **D**



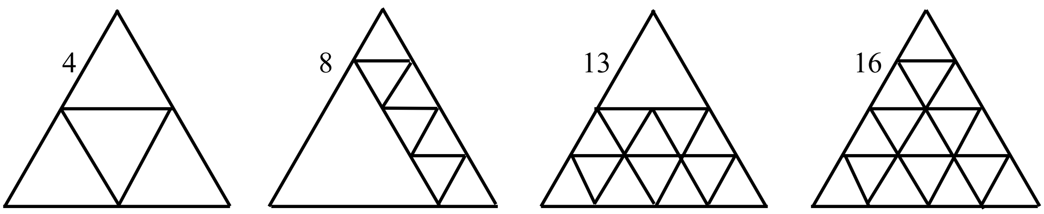
The arrows show all the routes from the initial square to the final square following the rules. The number in the bottom right corner of each square shows the number of ways of reaching it following these arrows. For the initial square this number is 1. For each other square it is the sum of the numbers in the other squares for which there are arrows leading into the given square. As this gives 8 as the number in the final square, there are 8 routes through the maze.

14. An equilateral triangle of side length 4 cm is divided up into smaller equilateral triangles, all of which have side length equal to a whole number of centimetres. Which of the following cannot be the number of smaller triangles obtained?

A 4 B 8 C 12 D 13 E 16

Solution: **C**

In the context of the SMC, it is enough to see that all the other options are possible. This is shown in the following diagrams:



However, a complete mathematical solution requires a proof that option C is not possible.

We first note that the big triangle can only be divided into smaller triangles with side lengths 1cm, 2cm and 3cm. There is room just for one triangle of side length 3cm, and we see, from the second diagram above, that if we have such a triangle in our subdivision, then there isn't room for a triangle of side length 2cm. So the remainder of the subdivision must consist of 7 triangles with side length 1cm. This gives a subdivision into 8 smaller triangles.

We now consider the possible subdivisions into, say, a, triangles with side length 1cm and b triangle with side length 2cm. By considering the areas of these triangles and that of the large triangle, we have that

$$a + 4b = 16, \text{ where } a \text{ and } b \text{ are non-negative integers.} \qquad (1)$$

There are just 5 solutions of (1), as shown in the table. This shows that the only possible values for $a+b$ are 4, 7, 10, 13 and 16. So a subdivision into 12 smaller triangles is not possible. Subdivisions into 4, 7, 10, 13 and 16 smaller triangles are all achievable. The cases 4, 13 and 16 are shown above.

a	b	$a+b$
0	4	4
4	3	7
8	2	10
12	1	13
16	0	16

Extension Problems

14.1 Give diagrams to show the cases where $a+b=7$ and where $a+b=10$.

14.2 Consider the different ways of dividing an equilateral triangle with side length 5cm into smaller equilateral triangles whose side lengths are a whole number of centimeters. What are the possible values for the number of smaller triangles in subdivisions of this type?

15. The equation $x^2 + ax + b = 0$, where a and b are different, has solutions $x = a$ and $x = b$. How many such equations are there?

A 0 B 1 C 3 D 4 E an infinity

Solution: **B**

The equation $x^2 + ax + b = 0$ has the solutions $x = a$ and $x = b$ if and only if $x^2 + ax + b$ may be factorized as $(x-a)(x-b)$, that is, if and only if

$$x^2 + ax + b \equiv x^2 - (a+b)x + ab \qquad (1)$$

Now, equating the coefficients of x and the constant terms, we see that (1) holds if and only if

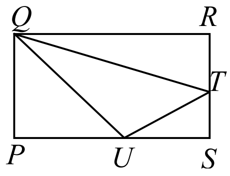
$$a = -(a+b) \qquad (2)$$

and

$$b = ab \qquad (3)$$

From (3) we have that either $a=1$ or $b=0$. If $a=1$ then, by (2), $b=-2$. If $b=0$, then by (2) $a=0$. Therefore, there is just one case, $a=1$, $b=-2$, where (1) holds and $a \neq b$.

16. $PQRS$ is a rectangle. The area of triangle QRT is $\frac{1}{5}$ of the area of $PQRS$, and the area of triangle TSU is $\frac{1}{8}$ of the area of $PQRS$. What fraction of the area of rectangle $PQRS$ is the area of the triangle QTU?



A $\dfrac{27}{40}$ B $\dfrac{21}{40}$ C $\dfrac{1}{2}$ D $\dfrac{19}{40}$ E $\dfrac{23}{60}$

Solution: **E**

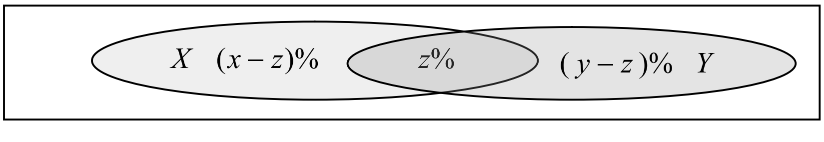
In the rectangle $PQRS$ we let $PS = QR = a$ and $PQ = SR = b$. Since the area of QRT is $\frac{1}{5}$ of the area of $PQRS$, we have that $\frac{1}{2}(a \times RT) = \frac{1}{5}(ab)$. Hence $RT = \frac{2}{5}b$, and therefore $TS = \frac{3}{5}b$. Since the area of TSU is $\frac{1}{8}$ of the area of $PQRS$, we have that $\frac{1}{2}(SU \times \frac{3}{5}b) = \frac{1}{8}ab$. Therefore $SU = \frac{5}{12}a$ and therefore $PU = \frac{7}{12}a$. Hence the area of the triangle PQU is $\frac{1}{2}(\frac{7}{12}a \times b) = \frac{7}{24}ab$. The area of the triangle QTU is the area of the rectangle $PQRS$ minus the areas of the triangles QRT, TSU and PQU, that is, $ab - \frac{1}{5}ab - \frac{1}{8}ab - \frac{7}{24}ab = \frac{23}{60}ab$. So the area of the triangle QTU is $\frac{23}{60}$ of the area of the rectangle $PQRS$.

17. Jamie conducted a survey on the food preferences of pupils at a school and discovered that 70% of the pupils like pears, 75% like oranges, 80% like bananas and 85% like apples. What is the smallest possible percentage of pupils who like all four of these fruits?

A at least 10% B at least 15% C at least 20% D at least 25% E at least 70%

Solution: **A**

We use the idea that if at least x% of the pupils are in a category X and at least y% are in a category Y, then at least $(x + y - 100)\%$ are in both categories. For suppose z% are in both categories. Then at least $(x - z)\%$ are in X but not Y, and at least $(y - z)\%$ in Y but not X. We must have that $(x - z)\% + z\% + (y - z)\% \le 100\%$, and hence $(x + y - 100)\% \le z\%$.



Now, as 70% of the pupils like pears and 75% like oranges, at least $(70 + 75 - 100)\% = 45\%$ like both pears and oranges. Hence, as 80% like bananas, at least $(45 + 80 - 100)\% = 25\%$ like pears, oranges and bananas. Therefore, as 85% like apples, at least $(25 + 85 - 100)\% = 10\%$ like all four fruits.

18. Two numbers x and y are such that $x + y = 20$ and $\frac{1}{x} + \frac{1}{y} = \frac{1}{2}$.

What is the value of $x^2 y + xy^2$?

A 80 B 200 C 400 D 640 E 800

Solution: **E**

We have that $\frac{1}{2} = \frac{1}{x} + \frac{1}{y} = \frac{x+y}{xy} = \frac{20}{xy}$. So $\frac{1}{2} = \frac{20}{xy}$ and hence $xy = 40$. Therefore

$x^2 y + xy^2 = xy(x + y) = 40 \times 20 = 800$.

Extension Problems

18.1 The polynomials $x + y$ and xy have the property that, in each case, if we interchange x and y the resulting polynomial is equivalent to the one we started with. That is, $y + x \equiv x + y$ and $yx \equiv xy$. For this reason they are called *symmetric polynomials.*

Which of the following are symmetric polynomials?

(a) $x^2 y + xy^2$, (b) $x^3 y^2 + x^2 y^2$, (c) $x^3 + y^3$, (d) $x^2 + y^2 + 3xy^2$.

18.2 It is a remarkable fact that every symmetric polynomial involving just x and y can be expressed in terms of the basic polynomials $x + y$ and xy. For example

$$x^2 + y^2 \equiv (x + y)^2 - 2xy. \qquad (1)$$

[To make this more precise we need to say what we mean by "expressed in terms of". Here this means that if $p(x, y)$ is a symmetric polynomial there is another polynomial, say, $P(X, Y)$, such that $p(x, y) \equiv P(x + y, xy)$. That is, we obtain $p(x, y)$ by substituting $x + y$ for X and xy for Y in the polynomial $P(X, Y)$. The example, (1), above shows that when $p(x, y)$ is the polynomial $x^2 + y^2$, we can take $P(X, Y)$ to be the polynomial $X^2 - 2Y$.]

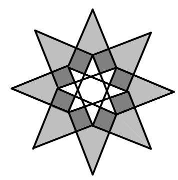
Express the following symmetric polynomials in terms of the polynomials $x + y$ and xy.

(a) $x^2 y + xy^2$, (b) $x^3 + y^3$, (c) $x^4 + y^4$, (d) $x^5 + y^5$.

18.3 The theory of symmetric polynomials applies also to polynomials involving more than two unknowns. For example, $x + y + z$, $xy + yz + zx$ and xyz are symmetric polynomials involving the unknowns x, y and z. They are symmetric because in each case, if we swap round the unknowns in any way we end up with a polynomial which is equivalent to the one we started with. For example, if we replace x by z, y by x and z by y, the polynomial $xy + yz + zx$ becomes $zx + xy + yz$ and $zx + xy + yz \equiv xy + yz + zx$. The polynomials $x + y + z$, $xy + yz + zx$ and xyz are the basic symmetric polynomials using the variables x, y and z. Every other symmetric polynomial using these unknowns can be expressed in terms of them.

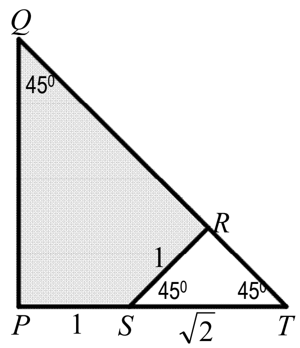
Express $x^3 + y^3 + z^3$ in terms of $x + y + z$, $xy + yz + zx$ and xyz.

19. The diagram shows a small regular octagram (an eight-sided star) surrounded by eight squares (dark grey) and eight kites (light grey) to make a large regular octagram. Each square has area 1. What is the area of one of the light grey kites?



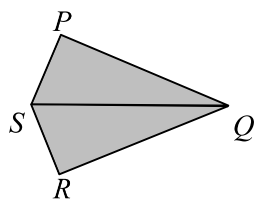
A 2 B $\sqrt{2}+1$ C $\frac{21}{8}$ D $4\sqrt{2}-3$ E $\frac{11}{4}$

Solution: **B**



We first give a geometrical solution, and then one which uses trigonometry.

Solution 1. We let *PQRS* be one of the kites. We suppose that the when the edges PS and QR are extended, they meet at the point *T*. As each of the squares has area 1, they have side length 1. So $PS = RS = 1$. The angle between the sides of a regular octagram is 45^0. So $\angle PQR = 45^0$. The angles $\angle QPS$ and $\angle QRS$ are right angles, so from triangle *PQT*, we have that $\angle RTS = 45^0$. Also, from the rectangle *PQRS*, $\angle PSR = 135^0$ and therefore $\angle RST = 45^0$. Hence the triangle RST is an isosceles right-angled triangle in which RS has length 1. So, by Pythagoras' Theorem, $ST = \sqrt{2}$. *PQT* is also a right-angled isosceles triangle. Hence, $PQ = PT = 1 + \sqrt{2}$. So the triangle PQS has base 1 and height $1+\sqrt{2}$, and hence area $\frac{1}{2}(1+\sqrt{2})$. Therefore the area of the kite, which is twice the area of the triangle *PQS*, is $1+\sqrt{2}$.



Solution 2. We let *PQRS* be the kite as in Solution 1. Since $\angle PQR = 45^0$ we have that $\angle PQS = \frac{1}{2}(45)^0$. Also $\angle QPS = 90^0$ and $PS = 1$, as it is the edge of a square which has area 1.

So the area of the triangle *PQS* is $\frac{1}{2}(1 \times PQ) = \frac{1}{2}PQ$. So the area of the kite which is twice the area of triangle *PQS* is *PQ*.

Now, from the right-angled triangle *PQS*, we have that $\frac{PS}{PQ} = \tan \angle PQS$ and therefore

$PQ = \frac{1}{\tan \frac{1}{2}(45)^0}$. From the formula $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$, and the fact that $\tan 45^0 = 1$, it follows, putting $\theta = \frac{1}{2}(45)^0$ and $x = \tan\frac{1}{2}(45)^0$, that $1 = \frac{2x}{1-x^2}$. Hence $1 - x^2 = 2x$, and therefore $x^2 + 2x - 1 = 0$. From the formula for the roots of a quadratic equation it follows that

$x = \dfrac{-2 \pm \sqrt{8}}{2} = -1 \pm \sqrt{2}$. Since $\tan \frac{1}{2}(45)^0 > 0$, it follows that $\tan \frac{1}{2}(45)^0 = -1 + \sqrt{2} = \sqrt{2} - 1$.

Therefore $PQ = \dfrac{1}{\sqrt{2}-1} = \dfrac{\sqrt{2}+1}{(\sqrt{2}-1)(\sqrt{2}+1)} = \sqrt{2}+1$. It follows that the area of the kite is $\sqrt{2}+1$.

Extension Problems

19.1 This problem is for students who are not familiar with the formula $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$ used in the above solution.

(a) Use the addition formulas $\sin(A+B) = \sin A \cos B + \cos A \sin B$ and $\cos(A+B) = \cos A \cos B - \sin A \sin B$, to obtain a formula for $\tan(A+B)$ in terms of $\tan A$ and $\tan B$.

(b) Use the formula you obtain for $\tan(A+B)$ to derive the formula above for $\tan 2\theta$.

19.2 Express $\tan 3\theta$ in terms of $\tan\theta$. Hence find a cubic equation with integer coefficients which has $\tan 15^0$ as one of its roots.

19.3 From the cubic equation which is the answer to 19.2 derive a quadratic equation with integer coefficients which has $\tan 15^0$ as one of its roots. Hence obtain an exact expression for $\tan 15^0$ in terms of surds.

20. Positive integers x and y satisfy the equation $\sqrt{x} - \sqrt{11} = \sqrt{y}$.

What is the maximum possible value of $\dfrac{x}{y}$?

A 2 B 4 C 8 D 11 E 44

Solution: **B**

We suppose that x and y are positive integers such that $\sqrt{x} - \sqrt{11} = \sqrt{y}$. As x and y are positive, we may square both sides, to give $x - 2\sqrt{11x} + 11 = y$. It follows that $2\sqrt{11x} = y - x - 11$ and so $2\sqrt{11x}$ is an integer. Suppose that $2\sqrt{11x} = n$, where n is an integer. Then, $4(11x) = n^2$ and so n is divisible by 2 and by 11. Hence for some positive integer a, $n = 22a$. Therefore $4(11x) = (22a)^2$ and therefore $x = 11a^2$. It follows that $y = x - 2\sqrt{11x} + 11 = 11a^2 - 22a + 11 = 11(a-1)^2$. Hence $\dfrac{x}{y} = \left(\dfrac{a}{a-1}\right)^2$.

For $a = 2$, we have that $\dfrac{a}{a-1} = 2$, and for $a \geq 3$, $\dfrac{a}{a-1} = 1 + \dfrac{1}{a-1} \leq 1 + \dfrac{1}{2} < 2$. So the maximum possible value of $\dfrac{x}{y}$ is $2^2 = 4$. This occurs when $a = 2$, and hence for $x = 44$ and $y = 11$.

21. Each of the Four Musketeers made a statement about the four of them, as follows:

d'Artagnan: "Exactly one is lying."

Athos: "Exactly two of us are lying."

Porthos: "An odd number of us is lying."

Aramis: "An even number of us is lying."

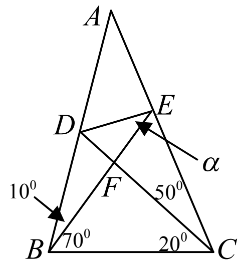
How many of them were lying (with the others telling the truth)?

A one B one or two C two or three D three E four

Solution: **C**

One of Porthos and Aramis is telling the truth, and the other is lying. Since they contradict each other, at least one of d'Artagnan and Athos is lying, and possibly both of them are lying. So it cannot be that just one person is lying or all four are lying. But it is possible either that two people (d'Artagnan and Porthos) are lying or that three of them (all but Porthos) are lying.

22. In the diagram, $\angle ABE = 10^0$; $\angle EBC = 70^0$; $\angle ACD = 50^0$; $\angle DCB = 20^0$; $\angle DEF = \alpha$.



Which of the following is equal to $\tan\alpha$?

A $\dfrac{\tan 10^0 \tan 20^0}{\tan 50^0}$ B $\dfrac{\tan 10^0 \tan 20^0}{\tan 70^0}$ C $\dfrac{\tan 10^0 \tan 50^0}{\tan 70^0}$

D $\dfrac{\tan 20^0 \tan 50^0}{\tan 70^0}$ E $\dfrac{\tan 10^0 \tan 70^0}{\tan 50^0}$

Solution: **A**

From triangle *BFC* we deduce that $\angle BFC = 90^0$. It follows that all the triangles *EFD*, *CFE*, *BFC* and *BFD* have a right angle at the vertex *F*. Therefore we have

$$\tan\alpha = \frac{DF}{EF},\ \tan 50^0 = \frac{EF}{CF},\ \tan 20^0 = \frac{BF}{CF} \text{ and } \tan 10^0 = \frac{DF}{BF}.$$

Therefore, $\tan\alpha = \dfrac{DF}{EF} = \dfrac{DF}{BF}.\dfrac{BF}{CF}.\dfrac{CF}{EF} = \tan 10^0 \times \tan 20^0 \times \dfrac{1}{\tan 50^0} = \dfrac{\tan 10^0 \tan 20^0}{\tan 50^0}$.

23. What is the minimum value of $x^2 + y^2 + 2xy + 6x + 6y + 4$?

A -7 B -5 C -4 D -1 E 4

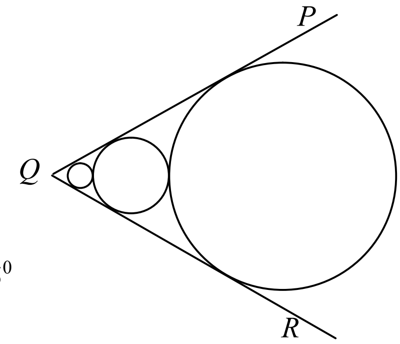
Solution: **B**

Note that the polynomial $x^2 + y^2 + 2xy + 6x + 6y + 4$ is symmetric in the sense described in the Extension Problems for Question 18.

We have that $x^2 + y^2 + 2xy + 6x + 6y + 4 = [x + y]^2 + 6[x + y] + 4 = ([x + y] + 3)^2 - 5$.

Since $([x + y]^2 + 3)^2 \geq 0$, it follows that $x^2 + y^2 + 2xy + 6x + 6y + 4 \geq -5$, and this value, -5, is achieved for $x + y + 3 = 0$, that is whenever $y = -x - 3$.

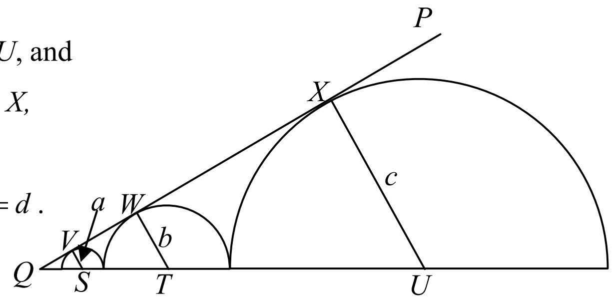
24. Three circles and the lines PQ and QR touch as shown. The distance between the centres of the smallest and biggest circles is 16 times the radius of the smallest circle. What is the size of $\angle PQR$?



A 45^0 B 60^0 C 75^0 D 90^0 E 135^0

Solution: **B**

We let the centres of the circles be S, T and U, and the points where they touch PQ be V, W and X, as shown. We let the radii of the circles be a, b and c, with $a < b < c$ as shown, let $QS = d$. and let $\angle PQU = \alpha$.



Then $QT = d + a + b$ and $QU = d + a + 2b + c$. Because PQ is a tangent to all three circles, the triangles QVS, QWT and QXU have right angles at V, W and X, respectively. Therefore

$$\sin\alpha = \frac{a}{d} = \frac{b}{d + a + b} = \frac{c}{d + a + 2b + c}.$$

Since $\frac{a}{d} = \sin\alpha$, we have $a = d\sin\alpha$.

Therefore, as $\frac{b}{d + a + b} = \sin\alpha$, we have $b = (d + a + b)\sin\alpha$. Hence $b - b\sin\alpha = (d + d\sin\alpha)\sin\alpha$ and therefore $b = \frac{d\sin\alpha(1 + \sin\alpha)}{1 - \sin\alpha}$.

Hence, as $\frac{c}{d + a + 2b + c} = \sin\alpha$, we have $c = (d + a + 2b + c)\sin\alpha$. Hence

$c - c\sin\alpha = \left(d + d\sin\alpha + \frac{2d\sin\alpha(1 + \sin\alpha)}{(1 - \sin\alpha)}\right)\sin\alpha$ and therefore

$$c = \frac{d\sin\alpha}{(1 - \sin\alpha)}\left(1 + \sin\alpha + \frac{2\sin\alpha(1 + \sin\alpha)}{(1 - \sin\alpha)}\right) = \frac{d\sin\alpha}{(1 - \sin\alpha)^2}\big((1 + \sin\alpha)(1 - \sin\alpha) + 2\sin\alpha(1 + \sin\alpha)\big)$$

$$= \frac{d\sin\alpha(1 + \sin\alpha)^2}{(1 - \sin\alpha)^2}.$$

Now, as the distance between the centres of the smallest and largest circles is 16 times the radius of the largest circle, $a + 2b + c = 16a$, and thus

$$d\sin\alpha + \frac{2d\sin\alpha(1+\sin\alpha)}{(1-\sin\alpha)} + \frac{d\sin\alpha(1+\sin\alpha)^2}{(1-\sin\alpha)^2} = 16d\sin\alpha\ .$$

We can divide through by $d\sin\alpha$ as this cannot be 0. We then obtain

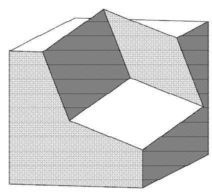
$$1 + 2\left(\frac{1+\sin\alpha}{1-\sin\alpha}\right) + \left(\frac{1+\sin\alpha}{1-\sin\alpha}\right)^2 = 1\ ,$$

that is,

$$\left(1+\left(\frac{1+\sin\alpha}{1-\sin\alpha}\right)\right)^2 = 16\text{, or, equivalently, }\left(\frac{2}{1-\sin\alpha}\right)^2 = 16\ .$$

It follows that $\frac{2}{1-\sin\alpha} = \pm 4$, giving $1-\sin\alpha = \pm\frac{1}{2}$, and hence that $\sin\alpha = \frac{1}{2}$, or $\sin\alpha = \frac{3}{2}$. The latter value is impossible, so $\sin\alpha = \frac{1}{2}$, and, as from the geometry of the problem, $0 < \alpha < 90^0$, we deduce that $\alpha = 30^0$. Therefore $\angle PQR = 2\alpha = 60^0$.

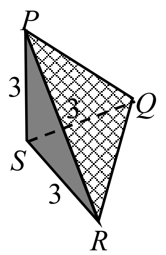
25. A solid sculpture consists of a $4\times4\times4$ cube with a $3\times3\times3$ cube sticking out as shown. Three vertices of the smaller cube lie on edges of the larger cube, the same distance along each edge. What is the total volume of the sculpture?



A 79 B 81 C 82 D 84 E 85

Solution: **C**

The volume of the sculpture is the sum of the volumes of the two cubes minus the volume of their intersection. This intersection consists of two tetrahedra which are both congruent to the tetrahedron



PQRS, shown, where *P*, *Q* and *R* are three vertices of the smaller cube. In one of these tetrahedra the fourth vertex *S* corresponds to the vertex of the smaller cube which is inside the larger cube. In the other, *S* corresponds to the vertex of the larger cube which is inside the smaller cube.

So *PQRS* is a tetrahedron whose base has area $b = \frac{1}{2}3^2 = \frac{9}{2}$ and with height $h = 3$. So the volume of *PQRS* is $\frac{1}{3}bh = \frac{9}{2}$. So the volume of the intersection which is made up of two of these tetrahedra is $2\times\frac{9}{2} = 9$.

It follows that the volume of the sculpture is $4^3 + 3^3 - 9 = 64 + 27 - 9 = 82$.