UK SENIOR MATHEMATICAL CHALLENGE

November 4th 2010

SOLUTIONS

These solutions augment the printed solutions that we send to schools. For convenience, the solutions sent to schools are confined to two sides of A4 paper and therefore in many cases are rather short. The solutions given here have been extended. In some cases we give alternative solutions, and we have included some ***Extension Problems*** for further investigations.

The Senior Mathematical Challenge (SMC) is a multiple choice contest, in which you are presented with five alternative answers, of which just one is correct. It follows that often you can find the correct answers by working backwards from the given alternatives, or by showing that four of them are not correct. This can be a sensible thing to do in the context of the SMC, and we often give first a solution using this approach.

However, this does not provide a full mathematical explanation that would be acceptable if you were just given the question without any alternative answers. So for each question we have included a complete solution which does not use the fact that one of the given alternatives is correct. Thus we have aimed to give full solutions with all steps explained. We therefore hope that these solutions can be used as a model for the type of written solution that is expected in the British Mathematical Olympiad and similar competitions.

We welcome comments on these solutions, and, especially, corrections or suggestions for improving them. Please send your comments,

either by e-mail to

enquiry@ukmt.co.uk

or by post to

SMC Solutions, UKMT Maths Challenges Office, School of Mathematics,
University of Leeds, Leeds LS2 9JT.

1. What is the digit x in this crossnumber?

1	2
3	x

Across
1. A cube
3. A cube

Down
1. One less than a cube

A 2 B 3 C 4 D 5 E 6

Solution: **C**

The only 2-digit cubes are $3^3 = 27$ and $4^3 = 64$. So the first column is either $\begin{array}{|c|}\hline 2 \\ \hline 2 \\ \hline\end{array}$, $\begin{array}{|c|}\hline 2 \\ \hline 6 \\ \hline\end{array}$, $\begin{array}{|c|}\hline 6 \\ \hline 2 \\ \hline\end{array}$ or $\begin{array}{|c|}\hline 6 \\ \hline 6 \\ \hline\end{array}$.

Of the possibilities for 1 down, only 26 is 1 less than a cube. So 3 across is 64 (and 1 across is 27). Hence, $x = 4$.

2. What is the smallest possible value of $20p + 10q + r$ when p, q and r are *different* positive integers?

A 31 B 43 C 53 D 63 E 2010

Solution: **B**

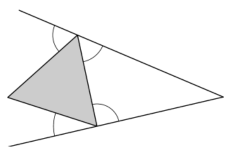
It seems "obvious" that p, q and r should be as small as possible, and, as the coefficient of p is larger than that of q, that we should take $p < q$, and, similarly, $q < r$. This suggests that the smallest value of $20p + 10q + r$, where p, q and r are different positive integers, is obtained by putting $p = 1$, $q = 2$, and $r = 3$, when we obtain the value $20 + 20 + 3 = 43$.

We can prove that this is the smallest possible value as follows. If $p \neq 1$, then $p \geq 2$, and hence, as $q > 0$, $20p + 10q + r \geq (20 \times 2) + (10 \times 1) = 50$. So the smallest possible value occurs when $p = 1$.

Now, if $q \neq 2$, $q \geq 3$, and so $20p + 10q + r \geq (20 \times 1) + (10 \times 3) = 50$. So the smallest possible value occurs when $p = 1$ and $q = 2$.

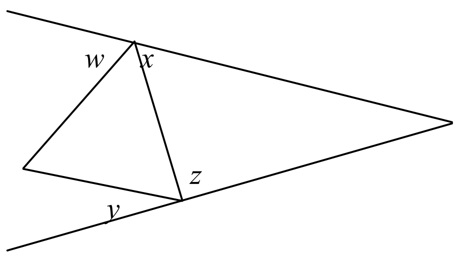
Finally, if $r \neq 3$, $r \geq 4$, and so $20p + 10q + r \geq (20 \times 1) + (10 \times 2) + 4 = 44$. So the smallest possible value is 43 and occurs when $p = 1$, $q = 2$ and $r = 3$.

3. The diagram shows an equilateral triangle touching two straight lines. What is the sum of the four marked angles?



A 120° B 180° C 240° D 300° E 360°

Solution: **C**



Let the angles be as marked. Since the angles in an equilateral triangle are each 60°, and the sum of the angles on a straight line is 180°, $w + x + 60^\circ = y + z + 60^\circ = 180^\circ$. Therefore, $w + x = y + z = 120^\circ$, and hence $w + x + y + z = 240^\circ$.

4. The year 2010 is one in which the sum of the digits is a factor of the year itself.
How many more years will it be before this is next the case?

A 3 B 6 C 9 D 12 E 15

Solution: **B**

In the context of the SMC, we need only check the given alternatives until we find one that is correct.

A. $2010+3=2013$. $2+0+1+3=6$. 6 is not a divisor of 2013.

B. $2010+6=2016$. $2+0+1+6=9$. 9 is a divisor of 2016.

To prove that this really is correct, we also need to check the other years from 2011 to 2015. We have that $2+0+1+1=4$ which is not a divisor of 2011, $2+0+1+2=5$ which is not a divisor of 2012, $2+0+1+4=7$ which is not a divisor of 2014, and $2+0+1+5=8$ which is not a divisor of 2015. Hence 2016 is the first year after 2010 with the required property.

5. A notice on Morecambe promenade reads 'It would take 20 million years to fill Morecambe Bay from a bath tap.' Assuming that the flow from the bath tap is 6 litres a minute, what does the notice imply is the approximate capacity of Morecambe Bay in litres?

A 6×10^{10} B 6×10^{11} C 6×10^{12} D 6×10^{13} E 6×10^{14}

Solution: **D**

We are only asked for an approximate answer, so we can ignore leap years. There are 60×24 minutes in a day, and hence $60\times24\times365$ minutes in a year, and hence $(60\times24\times365)\times(20\times10^6)$ minutes in 20 million years. Thus, according to the notice, the capacity of Morecambe Bay is approximately $(60\times24\times365)\times(20\times10^6)\times6$ litres. We now need to find an efficient way to approximate this number. We use the symbol "$\approx$" for "is approximately equal to".

We have that $24\times365\approx25\times360=25\times(4\times90)=(25\times4)\times90=100\times90=9000$. Therefore

$$(60\times24\times365)\times(20\times10^6)\times6\approx(60\times9000)\times(20\times10^6)\times6=(6\times9\times2\times6)\times(10\times1000\times10\times10^6)$$
$$=(54\times2\times6)\times10^{11}\approx(100\times6)\times10^{11}=6\times10^{13}.$$

Note: This estimate could be obtained in other ways and you may be able to find a more efficient method. The notice mentioned in the question may be found on the stretch of the promenade between the centre of Morecambe and Heysham. We don't know how the capacity of the Bay was measured.

Extension Problem: Check the plausibility of the estimated capacity by using a map to estimate the area of Morecambe Bay, to see what the estimated capacity implies about the average depth of the water.

6. Dean runs up a mountain road at 8 km per hour. It takes him one hour to get to the top. He runs down the same road at 12 km per hour. How many minutes does it take him to run down the mountain?

A 30 B 40 C 45 D 50 E 90

Solution: **B**

Since it takes 1 hour at 8 km per hour to get to the top, the length of the mountain road is 8km. Hence, running at 12 km per hour it takes $\frac{8}{12}$th of an hour, that is, 40 minutes, to run down the road.

Note: It isn't really necessary to work out that the length of the road is 8 km. We can argue directly that running at 12 km per hour takes $\frac{8}{12}$th of the time taken to run the same distance at 8 km per hour. So the time taken to run down the road is $\frac{8}{12}$th of the time taken to run up it.

7. There are 120 arrangements of the five letters in the word ANGLE. If all 120 are listed in alphabetical order starting with AEGLN and finishing with NLGEA, which position in the list does ANGLE occupy? A 18th B 20th C 22nd D 24th E 26th

Solution: **C**

There are $4! = 24$ arrangements of the 4 letters NGLE. In alphabetical order these run from EGLN to NLGE. So the first 24 arrangements of the letters in ANGLE run from AEGLN to ANLGE. Since there are $3! = 6$ arrangements of 3 letters, the first 6 of these 24 arrangements begin AE, the next 6 with AG, the next 6 with AL, and the final 6 with AN. So the 19th to 24th arrangements are ANEGL, ANELG, ANGEL, ANGLE, ANLEG, ANLGE. Thus ANGLE is the 22nd arrangement in the list.

8. Which of the following is equivalent to $(x+y+z)(x-y-z)$? A $x^2-y^2-z^2$ B $x^2-y^2+z^2$ C $x^2-xy-xz-z^2$ D $x^2-(y+z)^2$ E $x^2-(y-z)^2$

Solution: **D**

We should try to avoid having to expand the product $(x+y+z)(x-y-z)$, and, instead look for something better. A background idea in problems of this kind is that we might be able to exploit the standard factorization of the difference of two squares, that is, $a^2-b^2=(a+b)(a-b)$. With this in mind we see that

$$(x+y+z)(x-y-z)=(x+(y+z))(x-(y+z))=x^2-(y+z)^2.$$

A complete mathematical answer requires us to show that none of the other expressions is equivalent to $(x+y+z)(x-y-z)$. How can we do this? You mustn't fall into the trap of thinking that just because expressions look different, they are different. For example, we learn in trigonometry that the expressions $\cos 2\theta$ and $1-2\sin^2\theta$, which look different, really are the same.

What do we mean that two algebraic expressions are the same? We mean that they give the same values for all the (relevant) values of the variables. So we can show that two algebraic expressions are *different* by giving a single set of values of the variables for which the expressions take different values.

There are lots of ways of doing this here. If we put $x=0$, $y=1$, $z=2$ (always try a simple example first!), we have that $(x+y+z)(x-y-z)=-9$, and A $x^2-y^2-z^2=-5$; B $x^2-y^2+z^2=3$; C $x^2-xy-xz-z^2=-4$; D $x^2-(y+z)^2=-9$; and E $x^2-(y-z)^2=-1$. This shows that none of the expressions A, B, C and E is equivalent to $(x+y+z)(x-y-z)$. (Indeed, since the expressions A, B, C, D and E have different values when $x=0$, $y=1$ and $z=2$, we can deduce that none of these expressions are equivalent.)

Extension Problem: Show that x^3+y^3 is equivalent to exactly one of the following expressions.

(i) $(x-y)(x^2+xy+y^2)$, (ii) $(x-y)(x^2-xy+y^2)$,

(iii) $(x+y)(x^2+xy+y^2)$, (iv) $(x+y)(x^2-xy+y^2)$.

9. The symbol ◊ is defined by $x \lozenge y = x^y - y^x$. What is the value of $(2 \lozenge 3) \lozenge 4$?

A -3 B $-\frac{3}{4}$ C 0 D $\frac{3}{4}$ E 3

Solution: **D**

We have that $2 \lozenge 3 = 2^3 - 3^2 = 8 - 9 = -1$. Hence $(2 \lozenge 3) \lozenge 4 = (-1) \lozenge 4 = (-1)^4 - 4^{-1}$

$= 1 - \frac{1}{4} = \frac{3}{4}$.

10. A square is cut into 37 squares of which 36 have area 1 cm^2. What is the length of the side of the *original* square?

A 6 cm B 7 cm C 8 cm D 9 cm E 10 cm

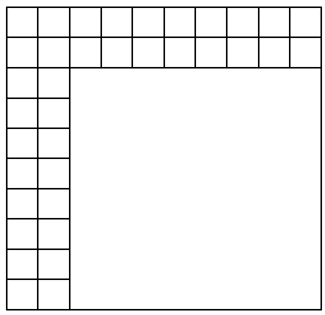
Solution: **E**

If the 37th square together with the 36 1×1 squares, makes up a square, both the 37th square, and the original square must have side lengths which are an integer number of cms.

So, if the side of the original square is x cm, we require that $x^2 - 36$ is also a square. In the context of the SMC, it is easy to check the given alternatives in turn. We have that $6^2 - 6^2 = 0$, $7^2 - 6^2 = 13$, $8^2 - 6^2 = 28$, $9^2 - 6^2 = 45$ and $10^2 - 6^2 = 64 = 8^2$. So we see that the answer is 10 cm.

An approach which does not work backwards from the given alternatives is to seek a solution of $x^2 - 36 = y^2$, with x and y positive integers. We have that $x^2 - 36 = y^2 \Leftrightarrow x^2 - y^2 = 36 \Leftrightarrow$ $(x+y)(x-y) = 36$. Since $x + y$, $x - y$ are positive integers with $x + y > x - y$, the only possibilities are $x + y = 36, x - y = 1$; $x + y = 18, x - y = 2$; $x + y = 12$, $x - y = 3$ and $x + y = 9, x - y = 4$. Only in the case $x + y = 18, x - y = 2$, are x and y integers, namely $x = 10$ and $y = 8$.

Finally, we need to check that 36 squares of area 1cm^2, can be fitted together with a square of side length 8, to make a square of side length 10. The diagram alongside shows one way this can be done. So $x = 10$ is the solution.



Note: We can see that only in the case $x + y = 18$, $x - y = 2$ are x and y integers, by directly solving each pair of equations. Alternatively, we could use a parity argument. If x and y are integers, then $x + y$, $x - y$ must either both be even numbers or both be odd numbers (why?). This rules out the cases $x + y = 36, x - y = 1$; $x + y = 12$, $x - y = 3$ and $x + y = 9, x - y = 4$.

11. What is the median of the following numbers?

A $9\sqrt{2}$ B $3\sqrt{19}$ C $4\sqrt{11}$ D $5\sqrt{7}$ E $6\sqrt{5}$

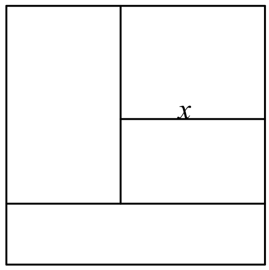
Solution: **D**

To arrange these numbers in order of magnitude it helps to calculate their squares. We see that

$(9\sqrt{2})^2 = 81 \times 2 = 162$; $(3\sqrt{19})^2 = 9 \times 19 = 171$; $(4\sqrt{11})^2 = 16 \times 11 = 176$; $(5\sqrt{7})^2 = 25 \times 7 = 175$; and $(6\sqrt{5})^2 = 36 \times 5 = 180$. Using the fact that for $x, y > 0$, if $x^2 < y^2$ then $x < y$, it follows that $9\sqrt{2} < 3\sqrt{19} < 5\sqrt{7} < 4\sqrt{11} < 6\sqrt{5}$. Hence the median of these numbers is $5\sqrt{7}$.

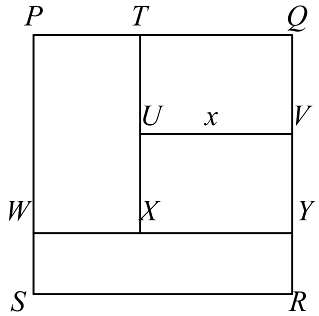
12. The diagram, which is not to scale, shows a square with side length 1, divided into four rectangles whose areas are equal. What is the length labelled *x*?

A $\frac{2}{3}$ B $\frac{17}{24}$ C $\frac{4}{5}$ D $\frac{49}{60}$ E $\frac{5}{6}$



Solution: **A**

We have labelled the vertices of the square *P, Q, R* and *S*, and the other points as shown.

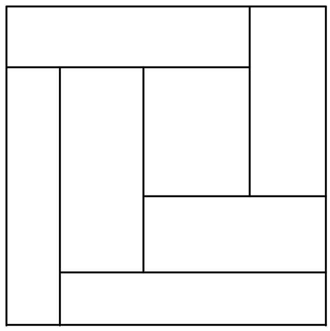


Each of the four rectangles into which the square is divided has the same area. So each rectangle has area $\frac{1}{4}$. So $WS = \frac{1}{4}$.

Hence $PW = \frac{3}{4}$. Hence, as the rectangle $PTXW$ has area $\frac{1}{4}$, $WX = \frac{1}{3}$. Hence $x = UV = XY = 1 - WX = 1 - \frac{1}{3} = \frac{2}{3}$.
[There are several other routes to this conclusion.]
.

Note: In this case the rectangles *TQVU* and *UVYX* are congruent. The background to this question is the problem of dividing a square into rectangles of equal area, no two of which are congruent.

The simplest solution involves dividing a square into 7 non-congruent rectangles with equal areas, as shown on the right.

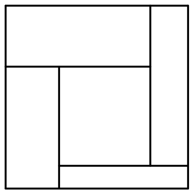


This result is due to Blanche Descartes, *Eureka*, 1971. *Blanche Descartes* was a collaborative pseudonym used by R. Leonard Brooks, Arthur Stone, Cedric Smith, and W. T. Tutte, who met in 1935 as undergraduate students in Cambridge. They proved a number of results about tessellations. Most notably, they solved the problem of squaring the square by showing that a square may be divided into smaller squares, no two of which are the same.

Extension Problems

(1) Show that it is not possible to divide a square into 2, 3 or 4 non-congruent rectangles with equal areas.

(2) Show that if a square is divided into 5 rectangles, with equal areas, as shown on the right, then at least two of the rectangles will be congruent.



(3) [*Hard*] Show that it is not possible to divide a square into 5 or 6 non-congruent rectangles with equal areas.

(4) [*Hard*] Calculate the dimensions of the rectangles in the division of a square into 7 rectangles as shown above (assuming that the square has side length 1).

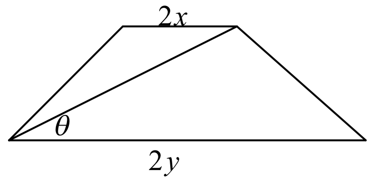
13. How many two-digit numbers have remainder 1 when divided by 3 and remainder 2 when divided by 4?

A 8 B 7 C 6 D 5 E 4

Solution: **A**

The positive integers that have remainder 1 when divided by 3 are 1, 4, 7, 10, 13, … and those that have remainder 2 when divided by 4 are 2, 6, 10, 14, …. The least positive integer in both these lists is 10. Now n is an integer which has the same remainder as 10 when divided by 3 and when divided by 4, if and only if $n-10$ is divisible by both 3 and 4. That is, if and only if $n-10$ is divisible by 12. So the integers that have remainder 1 when divided by 3, and remainder 2 when divided by 4, are those of the form $12k+10$, where k is an integer. The two-digit positive integers of this form are 10, 22, 34, 46, 58, 70, 82 and 94. There are 8 numbers in this list.

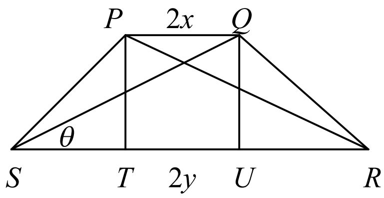
14. The parallel sides of a trapezium have lengths $2x$ and $2y$ respectively. The diagonals are equal in length, and one diagonal makes an angle θ with the parallel sides as shown. What is the length of each diagonal?



A $x+y$ B $\dfrac{x+y}{\sin\theta}$ C $(x+y)\cos\theta$ D $(x+y)\tan\theta$ E $\dfrac{x+y}{\cos\theta}$

Solution: **E**

We label the vertices of the trapezium *P, Q, R* and *S* as shown, and we let *T, U* be the points where the perpendiculars from *P, Q*, respectively, to the line *RS* meet this line.



Since the diagonals *SQ* and *PR* are equal and $PT = QU$, we see, that the right-angled triangles QUS and PTR are congruent. So $SU = TR$ and hence $ST = UR$. Now $ST + UR = SR - TU$ $= SR - PQ = 2y - 2x$. Hence $ST = y - x$ and therefore $SU = ST + TU = ST + PQ = (y-x) + 2x = x + y$. From the right-angled triangle QSU we have $\dfrac{SU}{SQ} = \cos\theta$. Hence $SQ = \dfrac{SU}{\cos\theta} = \dfrac{x+y}{\cos\theta}$.

15. What is the smallest prime number that is equal to the sum of two prime numbers and is also equal to the sum of three different prime numbers?

A 7 B 11 C 13 D 17 E 19

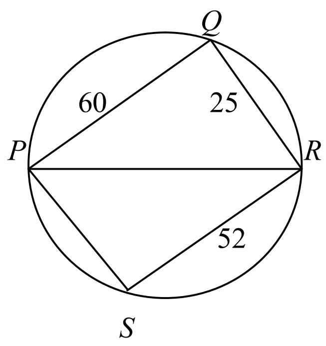
Solution: **E**

The sum of two odd prime numbers will be an even number greater than 2, and so cannot be a prime. So the only way for a prime number to be the sum of two prime numbers is for it to be of the form $2+p$, where p is a prime number. Similarly, the sum of three prime numbers can only be prime if it is the sum of three odd primes. The three smallest odd prime numbers are 3, 5, 7, but $3+5+7=15$, which is not prime. The next smallest sum of three odd primes is $3+5+11=19=2+17$. So 19 is the smallest prime number which is both the sum of two prime numbers and the sum of three different prime numbers.

16. *PQRS* is a quadrilateral inscribed in a circle of which *PR* is a diameter. The lengths of *PQ*, *QR* and *RS* are 60, 25 and 52 respectively. What is the length of *SP*?

A $21\frac{2}{3}$ B $28\frac{11}{13}$ C 33 D 36 E 39

Solution: **E**



Because PR is a diameter, the angles $\angle PQR$ and $\angle PSR$ are both right angles. Therefore, by Pythagoras' Theorem $PR^2 = PQ^2 + QR^2$ and $PR^2 = PS^2 + RS^2$, and hence $PQ^2 + QR^2 = RS^2 + SP^2$.

Therefore, $SP^2 = PQ^2 + QR^2 - RS^2 = 60^2 + 25^2 - 52^2$.
We now need to find the value of *SP* without using a calculator.
We have that $60^2 + 25^2 = (5\times 12)^2 + (5\times 5)^2 = 5^2(12^2 + 5^2) = 5^2(144+25) = 5^2 \times 13^2 = (5\times 13)^2$.

Therefore $SP^2 = 60^2 + 25^2 - 52^2 = (5\times 13)^2 - (4\times 13)^2 = (5^2 - 4^2)\times 13^2 = 3^2 \times 13^2 = (3\times 13)^2$.

It follows that $SP = 39$.

Notes:

1) You should already know that the angle in a semi-circle is a right angle.

***Extension Problem*:** Can you prove this?

2) We have that $PR^2 = PQ^2 + QR^2 = 60^2 + 25^2 = 3600 + 625 = 4225 = 65^2$. So the side lengths of the right angled triangle *PQR* are 25, 60 and 65. Note that these are in the ratio 5:12:13. The numbers 5, 12, 13 form what is called a *primitive Pythagorean triple*, that is they are integer solutions of the equation $x^2 + y^2 = z^2$, where *x*, *y* and *z* have no common factors other than 1.

Extension Problem: Find the primitive Pythagorean triple which corresponds in a similar way to the side lengths of triangle *PRS.*

17. One of the following is equal to $\sqrt{9^{16x^2}}$ for all values of *x*. Which one?

A 3^{4x} B 3^{4x^2} C 3^{8x^2} D 9^{4x} E 9^{8x^2}

Solution: **E**

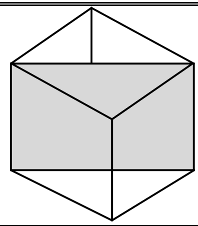
We use the fact that $\sqrt{a} = a^{\frac{1}{2}}$ when $a > 0$, and the index rule $(a^b)^c = a^{bc}$. It follows that

$\sqrt{9^{16x^2}} = (9^{16x^2})^{\frac{1}{2}} = 9^{(16x^2 \times \frac{1}{2})} = 9^{8x^2}$.

To complete the solution we need to show that none of the other expressions is equal to $\sqrt{9^{16x^2}}$ for all values of *x*. As in the solution to Question 8, it is good enough to find a single value of *x* for which expressions A, B, C and D, do not have the same value as $\sqrt{9^{16x^2}}$ When $x = 1$, $\sqrt{9^{16x^2}} = 9^8$ while the values of A, B, C and D are 3^4, 3^4, 3^8 and 9^4, respectively. This shows that none of them is equal to $\sqrt{9^{16x^2}}$ for all values of *x*.

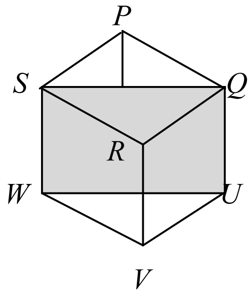
***Extension Problem**:* Find a value of *x* to show that the expressions A and B are not equivalent, and a value of *x* to show that expressions C and D are not equivalent.

18. A solid cube of side 2 cm is cut into two triangular prisms by a plane passing through four vertices as shown. What is the total surface area of these two prisms?



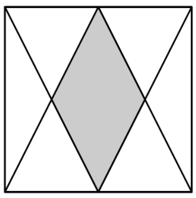
A $8(3+\sqrt{2})$ B $2(8+\sqrt{2})$ C $8(3+2\sqrt{2})$ D $16(3+\sqrt{2})$ E $8\sqrt{2}$

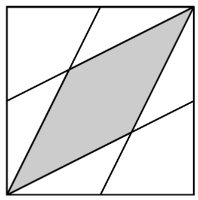
Solution: **A**



We label the vertices of the cube as shown. The total surface area of the two prisms is the surface area of the cube plus twice the area of *SQUW*, as this is a face of both prisms.
Each of the six faces of the cube has side 2 cm and so has area.4 cm^2.
So the surface area of the cube is $6 \times 4\,\text{cm}^2 = 24\,\text{cm}^2$.
By Pythagoras' Theorem, SQ and WU each have length $2\sqrt{2}$ cm.
Hence the rectangle $SQUW$ has area $4\sqrt{2}$ cm^2.
So the total surface area of the two prisms is $24 + 2 \times 4\sqrt{2}$ cm^2 =
$24 + 8\sqrt{2}$ cm^2 $= 8(3+\sqrt{2})$ cm^2.

19. The diagrams show two different shaded rhombuses each inside a square with sides of length 6.



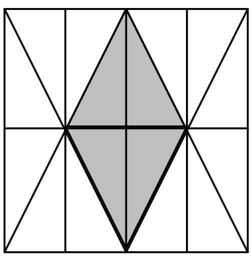


Each rhombus is formed by joining vertices of the square to midpoints of the sides of the square. What is the difference between the shaded areas?

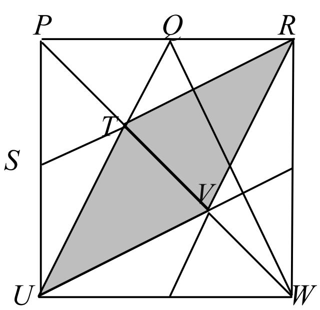
A 4 B 3 C 2 D 1 E 0

Solution: **B**

The square has side 6 and hence has area 36.



By adding three vertical lines and one horizontal line, as shown, we divide the square into 16 congruent triangles. The shaded rhombus is made up of 4 of these triangles. So the shaded area is one quarter of the total area of the square. So the shaded rhombus has area 9.



We add the diagonal from the top-left vertex to the bottom-right vertex of the square, and label the points *P, Q, R, S, T, U, V* and *W* as shown. The triangles *QRT, PQT, PTS* have bases *PQ, QR, PS* respectively of equal lengths, and they have equal heights. So they have the same area. They make up the triangle *PRS* whose area is one-quarter of the area of the square. So each of the triangles *QRT, PQT* and *PTS* has area one-twelfth that of the square, namely 3. So triangle *PTR* has area 6. Triangle *RVW* is congruent to triangle *PTR* and so also has area 6. Now triangle *PRW* has area 18, and hence the area of triangle *RTV* is $18 - 6 - 6 = 6$. Similarly, triangle *TUV* has area 6. Hence the area of the shaded rhombus is 12.
It follows that the difference in the areas of the shaded rhombuses in the two figures is $12 - 9 = 3$.

20. There are 10 girls in a mixed class. If two pupils from the class are selected at random to represent the class on the School Council, then the probability that both are girls is 0.15. How many boys are in the class?

A 10 B 12 C 15 D 18 E 20

Solution: **C**

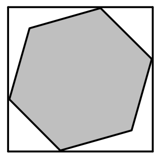
Suppose there are x boys in the class. Then there are $x+10$ pupils altogether, of whom 10 are girls. So the probability that the first pupil chosen is a girl is $\frac{10}{x+10}$. If a girl is chosen, there remain $x+9$ pupils, of whom 9 are girls, so that the probability that the second pupil chosen is also a girl is $\frac{9}{x+9}$.

Hence the probability that the two chosen pupils are both girls is $\frac{10}{10+x} \times \frac{9}{9+x} = \frac{90}{(10+x)(9+x)}$.

Therefore, we need to solve the equation $\frac{90}{(x+10)(x+9)} = 0.15$. In the context of the SMC you could just try the suggested alternatives in turn. A mathematically better method is to solve this equation algebraically. We have that $\frac{90}{(10+x)(9+x)} = 0.15 \Leftrightarrow 90 = \frac{3}{20}(10+x)(9+x)$

$\Leftrightarrow 600 = x^2 + 19x + 90 \Leftrightarrow x^2 + 19x - 510 = 0 \Leftrightarrow (x+34)(x-15) = 0$. This gives $x = 15$ or -34. Since you shouldn't have negative pupils (and certainly you can't have a negative number of them), the solution is $x = 15$.

21. The diagram shows a regular hexagon, with sides of length 1, inside a square. Two vertices of the hexagon lie on a diagonal of the square and the other four lie on the edges.



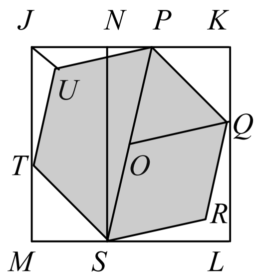
What is the area of the square?

A $2+\sqrt{3}$ B 4 C $3+\sqrt{2}$ D $1+\frac{3\sqrt{3}}{2}$ E $\frac{7}{2}$

Solution: **A**

We give three alternative solutions to this problem.

(1) Suppose that the square has side length y. We let J, K, L and M be the vertices of the square and let P, Q, R, S, T and U be the vertices of the hexagon, as shown. We let O be the centre of the hexagon, and N be the point where the perpendicular from S to JK meets JK.



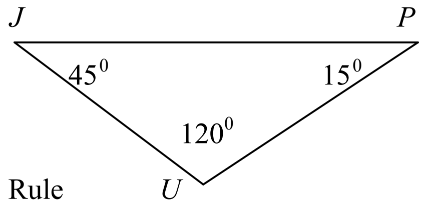
Since PQ is parallel to the diagonal JL, $\angle QPK = 45^0$. The triangle PQO is equilateral. Hence $\angle QPO = 60^0$. It follows that $\angle SPN = 180^0 - 45^0 - 60^0 = 75^0$. $SNJM$ is a rectangle and hence $SN = MJ = y$. Also $PO = OS = PQ = 1$, and hence $PS = 2$.

From the right angled triangle PSN, we have that $\sin 75^0 = \frac{SN}{PS} = \frac{y}{2}$. Therefore

$y = 2\sin 75^0 = 2\sin(30^0 + 45^0) = 2[\sin 30^0 \cos 45^0 + \cos 30^0 \sin 45^0] = 2[\frac{1}{2}.\frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2}.\frac{1}{\sqrt{2}}] = \frac{1}{\sqrt{2}}(1+\sqrt{3})$.

It follows that $y^2 = \frac{1}{2}(1+\sqrt{3})^2 = \frac{1}{2}(1+2\sqrt{3}+3) = 2+\sqrt{3}$.

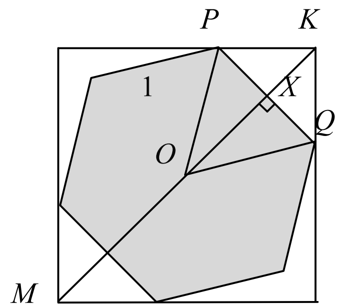
(2) We apply the Sine Rule to the triangle *JPU.*



In this triangle, $\angle PJU = 45^0$. $\angle JPU = 180^0 - 45^0 - 120^0 = 15^0$ and hence $\angle PUJ = 180^0 - 45^0 - 15^0 = 120^0$. We also have that $JP = JK - KP = y - \frac{1}{\sqrt{2}}$ and $PU = 1$. Therefore, applying the Sine Rule to the triangle *JPU*, we obtain $\dfrac{PU}{\sin\angle PJU} = \dfrac{JP}{\sin\angle PUJ}$ that is $\dfrac{1}{\sin 45^0} = \dfrac{y - \frac{1}{\sqrt{2}}}{\sin 120^0}$ and hence

$y - \dfrac{1}{\sqrt{2}} = \dfrac{\sin 120^0}{\sin 45^0} = \dfrac{\sin 60^0}{\sin 45^0} = \dfrac{\sqrt{3}}{2}\Big/\dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{3}}{\sqrt{2}}$. Therefore $y = \dfrac{1}{\sqrt{2}} + \dfrac{\sqrt{3}}{\sqrt{2}} = \dfrac{1}{\sqrt{2}}(1+\sqrt{3})$, and, as in the first method, $y^2 = 2 + \sqrt{3}$.

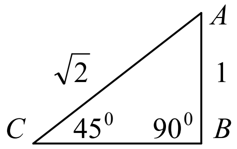
(3) Our third method avoids the use of trigonometry. We let *K, M, O, P* and *Q* be as in the diagram of solution (1). We join the common centre, *O*, of the square and the hexagon, to the points *K, M, P* and *Q*. Let *X* be the point where *KM* meets *PQ.* Then *OX* is the height of the equilateral triangle *OPQ* and *XK* is the height of the isosceles right-angled triangle *KPQ*.



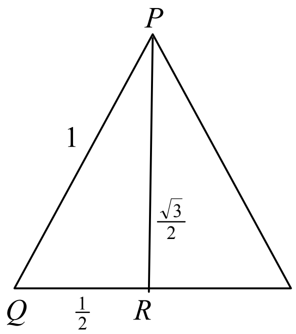
Now $OQ = 1$ and $QX = \frac{1}{2}$. So, applying Pythagoras' Theorem to triangle OXQ, we have $OX = \frac{1}{2}\sqrt{3}$. We also have $XK = XQ = \frac{1}{2}$.

Hence $KM = 2OK = 2(OX + XK) = \sqrt{3} + 1$. It follows that the side length of the square is $\frac{1}{\sqrt{2}}KM = \frac{1}{\sqrt{2}}(\sqrt{3}+1)$, and hence the area of the square is $\left(\frac{1}{\sqrt{2}}(\sqrt{3}+1)\right)^2 = 2 + \sqrt{3}$.

Note: The first two methods require us to know (*without a calculator*) the sines and cosines of the angles. 30^0, 45^0 and 60^0. These can be remembered using the right angled isosceles triangle with angles 45^0, 45^0 and 90^0, and the triangle with degrees 30^0, 60^0 and 90^0 obtained by bisecting an equilateral triangle.



If the sides, *AB, BC,* adjacent to the right angle have length 1, then, by Pythagoras' Theorem, the hypotenuse, *AC*, has length $\sqrt{2}$. Therefore $\sin 45^0 = \dfrac{AB}{AC} = \dfrac{1}{\sqrt{2}}$ and $\cos 45^0 = \dfrac{BC}{AC} = \dfrac{1}{\sqrt{2}}$.



If *PQ* has length 1, then $QR = \dfrac{1}{2}$. Therefore by Pythagoras' Theorem applied to the right-angled triangle *PQR*, we have that $\left(\dfrac{1}{2}\right)^2 + PQ^2 = 1^2$, and hence $PQ^2 = 1 - \dfrac{1}{4} = \dfrac{3}{4}$, giving, $PR = \dfrac{\sqrt{3}}{2}$.

We have from triangle *PQR* that $\sin 30^0 = \cos 60^0 = \dfrac{QR}{PQ} = \dfrac{1}{2}$ and $\sin 60^0 = \cos 30^0 = \dfrac{PR}{PQ} = \dfrac{\sqrt{3}}{2}$.

Extension Problems:

1. Find expressions for $\sin 15^0$ and $\cos 15^0$ in terms of surds.
2. Find expressions for $\sin 7.5^0$ and $\cos 7.5^0$ in terms of surds.

22. If $x^2 - px - q = 0$, where p and q are positive integers, which of the following could not equal x^3?

A $4x+3$ B $8x+5$ C $8x+7$ D $10x+3$ E $26x+5$

Solution: **B**

If $x^2 - px - q = 0$, then $x^2 = px + q$. Hence $x^3 = px^2 + qx = p(px+q) + qx = (p^2+q)x + pq$.

The following table shows that options A, C, D and E are possible.

p	q	$(p^2+q)x+pq$
1	3	$4x+3$
1	7	$8x+7$
3	1	$10x+3$
5	1	$26x+5$

To complete the proof we need to show that option B is not possible. We do this by showing that the assumption that $x^3 = 8x+5$ leads to a contradiction.

Suppose $x^3 = 8x+5$. Then $(p^2+q)x + pq = 8x+5$. It follows that either

$$pq = 5 \text{ and } p^2 + q = 8, \text{ or } x = \frac{5-pq}{p^2+q-8}, \text{ where } p^2+q-8 \neq 0. \qquad (1)$$

If $pq = 5$, then either $p=1$ and $q=5$, giving $(p^2+q)x+pq = 6x+5$, or $p=5$ and $q=1$, giving $(p^2+q)x+pq = 26x+5$. So the first alternative of (1) is not possible. We now need to rule out the second alternative. As *p* and *q* are integers, in this case *x* is a rational number. Therefore it will be sufficient to show that the cubic equation $x^3 = 8x+5$ has no rational solutions. We do this using a standard method.

Suppose that the equation $x^3 = 8x+5$ has a rational solution. We can suppose this solution has the form r/s, where *r, s* are integers with no common factor, and $s > 0$. Since this is a solution of the equation, we have

$$\left(\frac{r}{s}\right)^3 = 8\frac{r}{s} + 5$$

and hence $$r^3 = 8rs^2 + 5s^3, \qquad (2)$$

or, equivalently $$r^3 - 8rs^2 = 5s^3 \qquad (3)$$

The left hand side of (3) is divisible by *r*. It follows that $5s^3$ is divisible by *r*. However *r* and *s* have no common factors. Therefore 5 must be divisible by *r*, and so *r* can only be 1, -1, 5 or -5.

The right hand side of (2) is divisible by *s*. It follows that r^3 is divisible by *s*. However *r* and *s* have no common factors and $s > 0$. Therefore $s = 1$.

Therefore the only possible values for r/s are 1, -1, 5 and -5. However it is easily checked that none of these is a solution of $x^3 = 8x+5$. So the second alternative of (1) is also not possible.

We therefore deduce that x^3 cannot equal $8x+5$.

Note: The method that we have used to show that the cubic equation $x^3 = 8x + 5$ has no solutions which are rational numbers can be generalized to give:

The Rational Root Test
If r/s is a rational solution of the polynomial equation with integer coefficients

$$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 = 0 ,$$

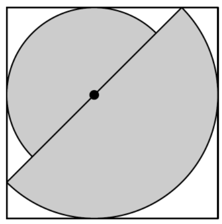
where r, s have no common factors, then r is a divisor of a_0 and s is a divisor of a_n

Try to prove this result. [This is really university level mathematics, and you will find a proof in many standard university textbooks, for example, R. B. J. T. Allenby, Numbers and Proofs, London, 1997, page 135.]

The advantage of the Rational Root Test is that the conditions that r is a divisor of a_0 and s is a divisor of a_n imply that there are only finitely many possible rational solutions, and these may be checked to determine whether or not the polynomial equation has a rational solution.

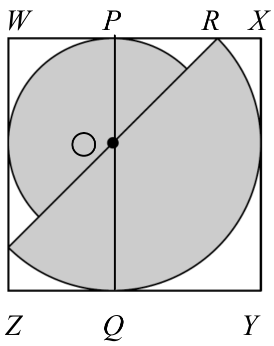
***Extension Problem**:*
Determine whether the following equations have rational roots.
a) $2x^5 + 3x^2 - 7 = 0$, b) $3x^4 + 4x^3 - 16x^2 + 5x + 2 = 0$, c) $x^7 + 3x^3 - 5 = 0$.

23. The diagram shows two different semicircles inside a square with sides of length 2. The common centre of the semicircles lies on a diagonal of the square.
What is the total shaded area?



A π B $6\pi(3 - 2\sqrt{2})$ C $\pi\sqrt{2}$ D $3\pi(2 - \sqrt{2})$ E $8\pi(2\sqrt{2} - 3)$

Solution: **B**



We let *W, X, Y, Z* be the vertices of the square, as shown. We let *O* be the common centre of the two semicircles, and we let *P, Q* be the points where the two semicircles touch the edges *WX* and *ZY*, respectively. Then as *OP* and *OQ* are radii of the semicircles, they are perpendicular to *WX* and *ZY*. So *POQ* is a straight line which is parallel to *WZ* and XY. We let *R* be the point where the larger semicircle meets *WX*, as shown.

Suppose that the smaller semicircle has radius r. From the right-angled isosceles triangle *POR* we see that the larger semicircle has radius $\sqrt{2}r$. Then $PQ = r + \sqrt{2}r$. Hence, as the square has side length 2, $r + \sqrt{2}r = 2$, and hence

$$r = \frac{2}{1+\sqrt{2}} = \frac{2(\sqrt{2}-1)}{(\sqrt{2}+1)(\sqrt{2}-1)} = 2(\sqrt{2}-1) .$$

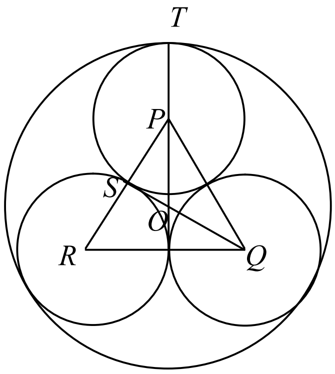
The shaded area is the sum of the areas of the two semicircles. So the shaded area is
$\frac{1}{2}\pi r^2 + \frac{1}{2}\pi(\sqrt{2}r)^2 = \frac{3}{2}\pi r^2 = \frac{3}{2}\pi(2(\sqrt{2}-1))^2 = \frac{3}{2}\pi \times 4(2 - 2\sqrt{2} + 1) = 6\pi(3 - 2\sqrt{2})$.

24. Three spheres of radius 1 are placed on a horizontal table and inside a vertical hollow cylinder of height 2 units which is just large enough to surround them. What fraction of the internal volume of the cylinder is occupied by the spheres?

A $\frac{2}{7+4\sqrt{3}}$ B $\frac{2}{2+\sqrt{3}}$ C $\frac{1}{3}$ D $\frac{3}{2+\sqrt{3}}$ E $\frac{6}{7+4\sqrt{3}}$

Solution: **E**

The diagram represents a horizontal cross section through the centres, say *P*, *Q*, *R*, of the spheres.



We let *O* be the point where this cross section meets the central axis of the cylinder. We let *S* be the point where the line *QO* meets *PR* and we let *T* be the point where the line *OP* meets the cylinder. We let $2x$ be the length of *OP*.

PQR is an equilateral triangle of side length 2. So *SPO* is a triangle with angles 90, 60 and 30 degrees, in which $PS = 1$, $OP = 2x$ and $SO = x$. By Pythagoras' Theorem applied to this triangle $1^2 + x^2 = (2x)^2$, and hence $3x^2 = 1$, so $x = 1/\sqrt{3}$. Hence the radius of the cylinder, which is equal to the length of *OT* is $OP + PT = \frac{2}{\sqrt{3}} + 1$.

It follows that the volume of the cylinder is $\pi \times 2 \times \left(\frac{2}{\sqrt{3}} + 1\right)^2 = 2\pi\left(\frac{4}{3} + \frac{4}{\sqrt{3}} + 1\right) = 2\pi\left(\frac{7}{3} + \frac{4}{\sqrt{3}}\right)$.

$= \frac{2\pi}{3}(7 + 4\sqrt{3})$. The total volume of the three spheres of radius 1 is $3\left(\frac{4}{3}\pi\right) = 4\pi$. Hence the ratio of these volumes is $\frac{4\pi}{\frac{2\pi}{3}(7+4\sqrt{3})} = \frac{6}{7+4\sqrt{3}}$. (Note that *O* is the *centroid* of the equilateral triangle *PQR*. It is a standard result that the centroid divides the medians of a triangle in the ratio 1:2.)

25. All the digits of a number are different, the first digit is not zero, and the sum of the digits is 36. There are $N \times 7!$ such numbers. What is the value of *N*?

A 72 B 97 C 104 D 107 E 128

Solution: **D**

$0 + 1 + 2 + ... + 9 = 45$, so we obtain a set of digits with sum 36 by omitting digits with sum 9. There are eight combinations of non-zero different digits with sum 9, namely 9, $8+1$, $7+2$, $6+3$, $5+4$, $6+2+1$, $5+3+1$ and $4+3+2$.

Now *k* digits can be arranged in order in $k!$ different ways. So, deleting 0 when it is the first digit, these *k* digits give rise to $k!$ different numbers, all with the same sum of digits. The numbers obtained for any set of digits will be different to those obtained from a different set, so we get the total number of such numbers, by adding up the number of numbers in each of the separate cases.

If we omit the digit 9, there remain 9 digits which form 9! different numbers whose digits have sum 36. Similarly, omitting 8,1 or 7,2 or 6,3 or 5,4, in each case we can form 8! different numbers whose digits have sum 36, and omitting 6,2,1 or 5,3,1 or 4,3,2, in each case we can form 7! such numbers. So the total number of such numbers is

$$9! + (4 \times 8!) + (3 \times 7!) = (9 \times 8 + 4 \times 8 + 3) \times 7! = (72 + 32 + 3) \times 7! = 107 \times 7!$$