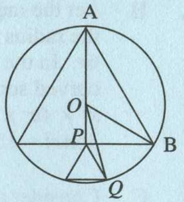
23. A Draw in the lines and attach labels as shown and let the circle have radius r. (O is the centre of the circle.) In ΔOPB: $OP = OB \sin 30° = \frac{1}{2}r$. Then, looking at ΔOPQ, $\angle OPQ = 150°$ so we can apply the Cosine Rule to obtain PQ.



$$OQ^2 = PQ^2 + OP^2 - 2PQ.OP \cos 150°$$

Writing $PQ = ry$, this gives

$$r^2 = r^2y^2 + \tfrac{1}{4}r^2 - 2 \times ry \times \tfrac{1}{2}r\left(-\tfrac{\sqrt{3}}{2}\right)$$

i.e. $$1 = y^2 + \tfrac{1}{4} + \tfrac{1}{2}\sqrt{3}y$$

$$\left(y + \tfrac{1}{4}\sqrt{3}\right)^2 = \tfrac{3}{4} + \left(\tfrac{\sqrt{3}}{4}\right)^2 = \tfrac{15}{16}$$

$$\therefore \quad y = \tfrac{\sqrt{15}}{4} - \tfrac{\sqrt{3}}{4} \quad \text{(ignoring the negative root)}.$$

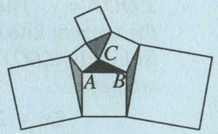
Hence the ratio $\dfrac{AB}{PQ} = \dfrac{\sqrt{3}r}{\left(\frac{\sqrt{15}-\sqrt{3}}{4}\right)r} = \dfrac{4}{(\sqrt{5} - 1)} \times \dfrac{\sqrt{5} + 1}{\sqrt{5} + 1} = \sqrt{5} + 1.$

24. E The possible combinations of Knaves who ate the tarts are shown in the table below. A tick opposite the letter of the statement means that the statement is consistent with that combination having eaten the tarts

	None	C	D	S	CD	DS	CS	CDS
A	✓	✗	✗	✗	✗	✗	✗	✗
B	✗	✓	✗	✗	✓	✗	✓	✓
C	✗	✓	✓	✓	✗	✗	✗	✗
D	✓	✓	✓	✓	✓	✗	✓	✗
E	✗	✗	✗	✗	✓	✓	✓	✓

The only column which contains exactly one tick is 'DS' in row E. Therefore E is the correct statement.

25. D Let the original triangle be ABC and use a, b, c for the opposite sides as normal. Then the sides of the three inner squares are a, b and c. Hence $A_1 = a^2 + b^2 + c^2$. We now move towards the three outer squares but first apply the Cosine Rule to the original triangle in three ways.



$$a^2 = b^2 + c^2 - 2bc \cos A \Rightarrow 2bc \cos A = b^2 + c^2 - a^2 \quad (1)$$

$$b^2 = a^2 + c^2 - 2ac \cos B \Rightarrow 2ac \cos B = a^2 + c^2 - b^2 \quad (2)$$

$$c^2 = a^2 + b^2 - 2ab \cos C \Rightarrow 2ab \cos C = a^2 + b^2 - c^2 \quad (3)$$

The area of the lower left square is given by

$$b^2 + c^2 - 2bc\cos(180° - A) = b^2 + c^2 + 2bc \cos A = 2(b^2 + c^2) - a^2 \text{ by (1)}.$$

The area of the lower right square is given by

$$a^2 + c^2 - 2ac\cos(180° - B) = a^2 + c^2 + 2ac \cos B = 2(a^2 + c^2) - b^2 \text{ by (2)}.$$

The area of the top square is given by

$$a^2 + b^2 - 2ab\cos(180° - C) = a^2 + b^2 + 2ab \cos C = 2(a^2 + b^2) - c^2 \text{ by (3)}.$$

Thus $$A_2 = 4(a^2 + b^2 + c^2) - (a^2 + b^2 + c^2) = 3A_1.$$

1.	C
2.	D
3.	D
4.	A
5.	D
6.	C
7.	E
8.	B
9.	B
10.	E
11.	A
12.	B
13.	D
14.	A
15.	A
16.	B
17.	C
18.	E
19.	C
20.	C
21.	C
22.	E
23.	A
24.	E
25.	D

UK SENIOR MATHEMATICAL CHALLENGE

Organised by the **United Kingdom Mathematics Trust**

SOLUTIONS

Keep these solutions secure until after the test on

TUESDAY 6 NOVEMBER 2001

This solutions pamphlet outlines a solution for each problem on this year's paper. We have tried to give the most straightforward approach, but the solutions presented here are not the only possible solutions. Occasionally we have added a 'Note' (in italics).

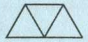
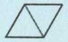
Please share these solutions with your students.

Much of the potential benefit of grappling with challenging mathematical problems depends on teachers making time for some kind of review, or follow-up, during which students may begin to see what they should have done, and how many problems they could have solved.

We hope that you and they agree that the first 15 problems could, in principle, have been solved by most candidates; if not, please let us know.

1. **C** 100 hours equal four days and four hours so the candle will burn out at 4pm on Thursday.

2. **D** Since 10^7 is not divisible by 11, then neither is $10^7 - 11$ nor $10^7 + 11$. Therefore, the correct answer is either B or D. $10^7 - 1 = 9\,999\,999 = 11 \times 909090 + 9$. Therefore $10^7 + 1 = 11 \times 909090 + 11$. Hence D is the correct answer.

3. **D** There are six quadrilaterals shaped  and six shaped .

4. **A** To have lost the fewest games, there need to be as many draws as possible. However, since 21 is odd, there needs to be at least one win. That is possible with 8 draws, leaving 11 games lost.

5. **D** Since the numbers offered are all odd, they need to be the sum of an even prime and an odd prime. But 2 is the only even prime so we need a number which is not two more than a prime. Since $13 = 2 + 11$; $33 = 2 + 31$; $43 = 2 + 41$ and $73 = 2 + 71$, we have $53 = 2 + 3 \times 17$ as the answer.

6. **C** Since the four numbers are symmetrical about the mean of the group, they must have the same mean, 21, giving their total as 84.
Alternatively, it is possible to state that the original numbers must be 15, 17, 19, 21, 23, 25, 27 (since the numbers are given as consecutive and odd). Hence the required sum is $15 + 19 + 23 + 27 = 84$.

7. **E** For each row that the apprentice plants, the gardener has planted one more in his previous row. The apprentice plants 50 rows and in addition the gardener plants the last one.

8. **B** Joining A to C and each of A, C and E to the centre of the hexagon divides it into six congruent triangles four of which make up $ABCE$. So the area of $ABCE$ is $\frac{2}{3}$ of 60.

9. **B** Since 6 and 9 are not prime, C and D cannot provide counter-examples. For 3, both parts are true (since 11 is prime) but since $5^2 + 2 = 27$ is not prime, B does provide a counter-example.

10. **E** There are six boxes adjacent to the box labelled **X**. However, each of the numbers 2, 3, 4, 5, 6 and 7 has only five numbers which are not adjacent to it and therefore the labelled box must contain 1 or 8.

11. **A** Since $x = \frac{1}{y}$, it follows that $y = \frac{1}{x}$. Hence $\left(x + \frac{1}{x}\right)\left(y - \frac{1}{y}\right) = (x + y)(y - x) = (y^2 - x^2)$.

12. **B** The shaded area is a right-angled isosceles triangle with height 1. The altitude divides it into two other right-angled isosceles triangles so the base of the shaded area is 2. Therefore the area is $\frac{1}{2} \times 2 \times 1 = 1$.

13. **D** Let the length of the first race be d and Rosie's average speed in this race be v (in appropriate units). Thus her time in this race was $\frac{d}{v}$.
The length of the second race was $d \times \frac{6}{5}$ and her average speed was $v \times \frac{4}{5}$. Hence her time for the second race was $\left(d \times \frac{6}{5}\right) \div \left(v \times \frac{4}{5}\right) = \frac{6}{5} \times \frac{5}{4} \times \frac{d}{v} = \frac{3}{2} \times \frac{d}{v}$. She took 50% longer.

14. **A** The altitude of an equilateral triangle of side 1 is $\frac{1}{2}\sqrt{3}$. Hence $XY = 2 \times \frac{1}{2}\sqrt{3} - 1 = \sqrt{3} - 1$.

15. **A** $5^8 \times 8^5 = 5^8 \times 2^{15} = (5 \times 2)^8 \times 2^7 = 128 \times 100\,000\,000 = 12\,800\,000\,000$ which has 11 digits.

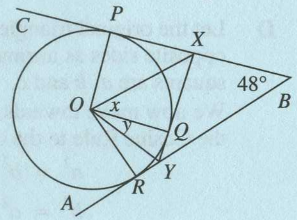
16. **B** Let the radius of the base of the top part of the cone (painted red) be r. So, by similarity, the radius of the red and white cone must be $4r$ and the radius of the whole cone must be $6r$. In the same way, the slant heights are l, $4l$ and $6l$. From the formula πrl for the curved surface area of a cone, it follows that the areas are πrl, $\pi \times 4r \times 4l$ and $\pi \times 6r \times 6l$. Thus the white area is $16\pi rl - \pi rl = 15\pi rl$ and the red area is $36\pi rl - 15\pi rl = 21\pi rl$ giving the ratio white area : red area $= 15\pi rl : 21\pi rl = 5 : 7$.

17. **C** Consider a group of 100 children, 10 from Scotland and 90 from elsewhere. Then using x to represent the required average we have $90x + 10 \times 5.35 = 100 \times 3.10$. Hence $90x = 310 - 53.5 = 256.5$ giving $x = 2.85$.

18. **E** The line $y = x - 4$ is closest to the origin at the point $(2, -2)$, a distance of $\sqrt{8} = 2\sqrt{2}$.
All the points on the circle $x^2 + y^2 = 4$ are 2 units from the origin.
The hyperbola $y = \frac{4}{x}$ is closest to the origin at its two nodes $(2, 2)$ and $(-2, -2)$, a distance of $\sqrt{8} = 2\sqrt{2}$.
The parabola $y = x^2 + 4$ has a local minimum at $(0, 4)$ so no point on it is closer than 4 units.
The curve $y = x^4 - 4$ cuts the x-axis at $(-\sqrt{2}, 0)$ and $(\sqrt{2}, 0)$, both of which are closer to the origin than any of the points on the other four alternatives.

19. **C** $2002 = 2 \times 1001 = 2 \times 7 \times 11 \times 13$ which is the product of four primes.
Since $2003 < 13 \times 13 \times 13$, for 2003 to be a product of three distinct primes, at least one of them must be less than 13. However, none of 2, 3, 5, 7 or 11 is a factor of 2003.
$2004 = 2 \times 2 \times 3 \times 167$ which has a repeated prime factor.
$2005 = 5 \times 401$ which is the product of two primes.
$2006 = 2 \times 1003 = 2 \times 17 \times 59$ and is therefore the next year which is the product of three distinct primes.

20. **C** Let the points of contacts of the tangents be P, Q and R as shown and let $\angle XOQ = x$ and $\angle QOY = y$. Then since OX is the axis of symmetry of the tangent kite $OPXQ$, it bisects $\angle POQ$ so $\angle XOP = x$. Similarly $\angle ROY = y$. Thus, in the quadrilateral $OPBR$ we have



$$2x + 2y + 2 \times 90° + 48° = 360°$$
i.e. $$2(x + y) = 132°$$
i.e. $$x + y = 66°$$

21. **C** Let the letters be p, q, r, s, t and the corresponding houses be P, Q, R, S, T.
The number of ways of correctly putting in two letters is $\frac{5!}{2!3!} = 10$. For the third letter, there are just two wrong choices and then the others are fixed. (If p, q have been correctly delivered, then clearly r can go to S or T. If r is put to S then t must go to R and s to T. If r is put to T then t must go to S and s to R.)
So there are just $2 \times 10 = 20$ ways.

22. **E** $4^{\left(\frac{n-1}{n+1}\right)}$ is an integer if the value of $\frac{n-1}{n+1}$ is $0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$ i.e. $\frac{n-1}{n+1} = \frac{k}{2}$ where $k = 0, 1, 2, 3, \ldots$
Let $\frac{k}{2} = \frac{n-1}{n+1} = \frac{n+1-2}{n+1} = 1 - \frac{2}{n+1}$. Then $k = 2 - \frac{4}{n+1}$.
For k to be an integer, the only possible values of $n + 1$ are $-4, -2, -1, 1, 2, 4$; i.e. the possible values of n are $-5, -3, -2, 0, 1, 3$.
The corresponding values of k are $3, 4, 6, -2, 0, 1$. However, k cannot be negative and therefore $n = 0$ is not valid.
Therefore the five possible values of n are $-5, -3, -2, 1, 3$.