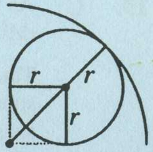
21. **A** B cannot be prime since 111 is clearly a factor of it; C cannot be prime since it equals 2999×1001 (difference of two squares); the units digit of D will be 5 and therefore 5 must be a factor of it whilst the units digit of E is 0 and therefore it cannot be prime either since both 2 and 5 will be factors of it.

22. **E** Let the radius of each large semicircle be R. Then $\pi R^2 = 4$. The circle in the left-hand diagram has radius $R/2$ and therefore its area is $\pi(R/2)^2 = (\pi R^2)/4 = 1$.
Let the radii of the circles in the right-hand diagram be r. Then $R = \sqrt{2}r + r$.



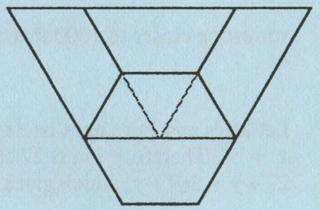
Therefore $r = \dfrac{R}{\sqrt{2}+1} = \dfrac{R(\sqrt{2}-1)}{(\sqrt{2}+1)(\sqrt{2}-1)} = R(\sqrt{2}-1)$.

The grey shaded area $= 2\pi r^2 = 2\pi R^2(\sqrt{2}-1)^2 = 2 \times 4 \times (2 - 2\sqrt{2} + 1) = 24 - 16\sqrt{2}$.

The difference in areas is therefore $23 - 16\sqrt{2}$.

23. **B** $$\frac{2n+y}{n-2} = \frac{2n-4}{n-2} + \frac{y+4}{n-2} = 2 + \frac{y+4}{n-2} \qquad (n \neq 2).$$
Thus $(2n + y)/(n - 2)$ is an integer if and only if $(y + 4)/(n - 2)$ is an integer. As n varies, the integer values taken by $(y + 4)/(n - 2)$ are all the integers which divide exactly into $y + 4$. There are exactly 4 of these if and only if $y + 4$ is prime. They are $\{(y + 4), -(y + 4), 1, -1\}$. Thus the integer values of the expression will be $2 + y + 4 = y + 6$; $2 - (y + 4) = -(y + 2)$; 3 and 1.
For $1 \leqslant y \leqslant 20$, the values of y for which $y + 4$ is prime are 1, 3, 7, 9, 13, 15 and 19.

24. **A** If the original tile is an isosceles trapezium made from three equilateral triangles then a quadrilateral (which is also an isosceles trapezium) will result.



25. **B** $$S = \frac{1}{2} + \frac{1}{4} + \frac{2}{8} + \frac{3}{16} + \frac{5}{32} + \frac{8}{64} + \frac{13}{128} + \frac{21}{256} + \frac{34}{512} + \dots$$
$$= \frac{1}{2} + \frac{1}{4} + \left(\frac{1}{8} + \frac{1}{8}\right) + \left(\frac{1}{16} + \frac{2}{16}\right) + \left(\frac{2}{32} + \frac{3}{32}\right) + \left(\frac{3}{64} + \frac{5}{64}\right) + \left(\frac{5}{128} + \frac{8}{128}\right) + \dots$$
$$= \frac{1}{2} + \left(\frac{1}{8} + \frac{1}{16} + \frac{2}{32} + \frac{3}{64} + \frac{5}{128} + \dots\right) + \left(\frac{1}{4} + \frac{1}{8} + \frac{2}{16} + \frac{3}{32} + \frac{5}{64} + \frac{8}{128} + \dots\right)$$
$$= \frac{1}{2} + \frac{1}{4}\left(\frac{1}{2} + \frac{1}{4} + \frac{2}{8} + \frac{3}{16} + \frac{5}{32} + \dots\right) + \frac{1}{2}\left(\frac{1}{2} + \frac{1}{4} + \frac{2}{8} + \frac{3}{16} + \frac{5}{32} + \frac{8}{64} + \dots\right)$$
$$= \frac{1}{2} + \frac{1}{4}S + \frac{1}{2}S = \frac{1}{2} + \frac{3}{4}S$$
Therefore $\frac{1}{4}S = \frac{1}{2}$ and so $S = 2$.

1.	C
2.	D
3.	D
4.	C
5.	B
6.	B
7.	D
8.	B
9.	C
10.	D
11.	E
12.	E
13.	C
14.	A
15.	E
16.	D
17.	C
18.	A
19.	C
20.	E
21.	A
22.	E
23.	B
24.	A
25.	B

UK SENIOR MATHEMATICAL CHALLENGE

Organised by the **United Kingdom Mathematics Trust**

SOLUTIONS

Keep these solutions secure until after the test on

TUESDAY 9 NOVEMBER 1999

This solutions pamphlet outlines a solution for each problem on this year's paper. We have tried to give the most straightforward approach, but the solutions presented here are not the only possible solutions. Occasionally we have added a 'Note' (in italics).

Please share these solutions with your students.

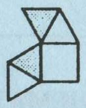
Much of the potential benefit of grappling with challenging mathematical problems depends on teachers making time for some kind of review, or follow-up, during which students may begin to see what they should have done, and how many problems they could have solved.

We hope that you and they agree that the first 15 problems could, in principle, have been solved by most candidates; if not, please let us know.

1. C The prime numbers less than 20 are 2, 3, 5, 7, 11, 13, 17, and 19.
(Prime numbers are defined as those numbers which have exactly two factors and therefore 1 is not prime since it has only one factor.)

2. D If January 1st (or, in a leap year, January 1st or January 2nd) falls on a Sunday, then there will be 53 Sundays in that particular year.

3. D The two shaded triangles will overlap when the 'net' is folded.



4. C The reductions would be A 20%; C $33\frac{1}{3}$ %; D 30%; E 25%. B might be a "special offer", but is not one which should be accepted!

5. B 17 hours and 21 minutes = 1041 minutes and the average speed is therefore slightly less than 40 decimal places per minute.

6. B The percentage increase = $\frac{500 - 40}{40} \times 100\% = \frac{460}{40} \times 100\% = 1150\%$.

7. D Let the exterior angle be $x°$. Then $x + 4x = 180$ and therefore $x = 36$.
As is the case in all convex polygons, the sum of the exterior angles = 360° and therefore the number of sides = 360/36 = 10.

8. B $x - y = 9$; $x + y = 99$. Adding gives $2x = 108$ and therefore $x = 54$ and hence $y = 45$.
The ratio $x{:}y = 54{:}45 = 6{:}5$.
(As we are interested only in the ratio rather than the numbers themselves, the problem could be reduced to finding the ratio of two numbers which differ by 1 and have sum 11.)

9. C All of the positive integers from 1 to 50 inclusive must be factors of 50!.
51 = 3 × 17 and 52 = 2 × 26 which means that these two numbers are also factors of 50!.
53 is prime and is not a factor of 50!.

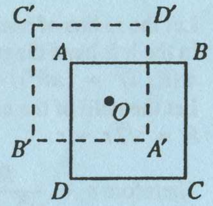
10. D The values are, approximately, A 45; B 1 × 30 = 30; C 20 × 10 = 200; D 200 × 3 = 600; E 2000 × 0 = 0.

11. E In the first row, any one of 5 letters could be circled. In the second row, any one of 4 letters could be circled since one column has now been occupied. Similarly, in the third row, any one of three letters could be circled and so on.
The number of different ways is therefore 5 × 4 × 3 × 2 × 1 = 120.
(Notice that this is 5! as in 9.)

12. E The White Rabbit must have been speaking on January 1st of this year. Two days earlier, December 30th, Alice was still thirteen and her fourteenth birthday was the following day, December 31st. She will, therefore, be fifteen on December 31st of this year and her sixteenth birthday will be on December 31st next year.

13. C As we are given, $AO = \frac{1}{3}AC$; therefore $AA' = \frac{2}{3}AC$ since $OA' = OA$.

Thus the area of overlap is the area of a square whose side is $\frac{2}{3}$ of the length of the side of square $ABCD$ i.e. the area of a square of side 2 cm.



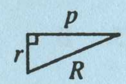
14. A The line $y = 3x + 4$ intersects the axes at $(-\frac{4}{3}, 0)$ and (0, 4). Its reflection in the line $y = -x$ will therefore intersect the axes at (–4, 0) and $(0, \frac{4}{3})$. The line through these points has gradient $\frac{1}{3}$ and therefore its equation is $y = \frac{1}{3}x + \frac{4}{3}$ or $3y = x + 4$.
(In general, the image of the point (a, b) after reflection in the line $y = -x$ is the point $(-b, -a)$.)

15. E $120 = 2^3 \times 3 \times 5$; $144 = 2^4 \times 3^2$; $240 = 2^4 \times 3 \times 5$; $3000 = 2^3 \times 3 \times 5^3$; $12100 = 2^2 \times 5^2 \times 11^2$. The product of the three numbers must have 6 prime factors, not necessarily all different, but with no prime factor repeated more than 3 times. Of these, only 12100 satisfies this condition. The three numbers are 10 (2 × 5), 22 (2 × 11) and 55 (5 × 11).

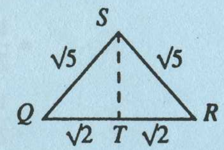
16. D x must be closer to 1000 than it is to 999 and also closer to 1000 than it is to 1010.

17. C Let Jon's and Jan's ages be $3x$ and x respectively. Then $3x - 3 = 4(x - 3)$ which gives $x = 9$. Therefore Jon is 27 and Jan is 9. If the ratio will be 2 : 1 in y years time then $27 + y = 2(9 + y)$ which gives $y = 9$. In 9 years time, Jon will be 36 and Jan will be 18.

18. A Let the radii of the outer and inner circles be R and r respectively. Then, by the Theorem of Pythagoras: $R^2 = r^2 + p^2$ and therefore $R^2 - r^2 = p^2$.
The area of the shaded region $= \pi R^2 - \pi r^2 = \pi(R^2 - r^2) = \pi p^2$.



19. C By the Theorem of Pythagoras, $QR = \sqrt{8}$ cm $= 2\sqrt{2}$ cm, $QS = \sqrt{5}$ cm and $RS = \sqrt{5}$ cm. If T is the midpoint of QR, then $TS^2 = (\sqrt{5})^2 - (\sqrt{2})^2 = 5 - 2 = 3$ and therefore $TS = \sqrt{3}$ cm. The area of triangle $QRS = \frac{1}{2}QR \times TS = (\sqrt{2} \times \sqrt{3})\ \text{cm}^2 = \sqrt{6}\ \text{cm}^2$.



20. E $1 + 2 + 3 + 4 + 5 + \ldots + n = \frac{1}{2}n(n + 1)$.
Therefore the terms in the sequence with positions $\frac{1}{2}(n - 1)n + 1$ to $\frac{1}{2}n(n + 1)$ inclusive will all be n. Now $\frac{1}{2} \times 62 \times 63 = 1953$ and $\frac{1}{2} \times 63 \times 64 = 2016$.
Therefore the 1954th term to the 2016th term inclusive will all be 63.