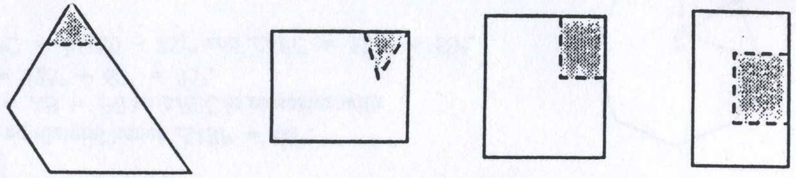
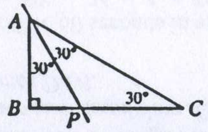


Note: the best way to see if $x \oplus y \rightarrow 4$ is possible is probably to start with a quadrilateral and test to see if you can form a polygon with y sides by removing a polygon with x sides.

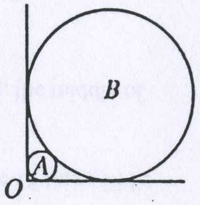
22. D



Points inside the triangle ABC which are closer to AB are to the left of AP. So the probability is the ratio of the area of ΔABP to the area of ΔABC which is the ratio of BP to BC since the triangles have the same height. BP to BC is equal to $\tan 30° : \tan 60°$ i.e. $1/\sqrt{3} : \sqrt{3} = 1 : 3$.

[Alternatively, draw the perpendicular from P to AC to show three congruent triangles.]

23. E The small circle can be enlarged to the size of the other circle by using a scale factor of R/r. So if A and B are the centres of the circles



$$\frac{R}{r} = \frac{OB}{OA} = \frac{r\sqrt{2} + r + R}{r\sqrt{2}}$$

$$\sqrt{2}R = r\sqrt{2} + r + R$$

$$\frac{R}{r} = \frac{\sqrt{2} + 1}{\sqrt{2} - 1} = \frac{(\sqrt{2} + 1)^2}{(\sqrt{2} - 1)(\sqrt{2} + 1)} = 3 + 2\sqrt{2}.$$

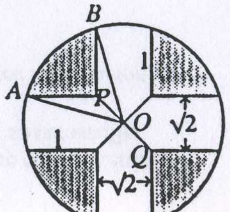
24. A This *could* be done by using the quadratic formula and manipulating surds. However, a neater way is

$$x^2 - 3x + 1 = 0 \Rightarrow x - 3 + 1/x = 0 \Rightarrow x + 1/x = 3.$$

Now, squaring both sides, $9 = (x + 1/x)^2 = x^2 + 2 + 1/x^2$. Hence $x^2 + 1/x^2 = 7$.

25. E $PQ^2 = (\sqrt{2})^2 + (\sqrt{2})^2 = 4 \Rightarrow PQ = 2 \Rightarrow OP = 1$.

Hence triangles APO and BPO are isosceles with $\angle APQ = \angle BPQ = 135°$. The area of each of these triangles is $\frac{1}{2} \times 1 \times 1 \times \sin 135° = 1/(2\sqrt{2})$. Also $\angle AOB = \angle AOP + \angle BOP = 45°$. The shaded area



$$APB = \text{sector } AOB - 2 \times (1/2\sqrt{2})$$

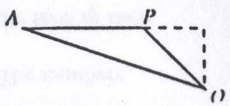
$$= \tfrac{1}{8}\pi OA^2 - 1/\sqrt{2}$$

$$= \tfrac{1}{8}\pi\,(1^2 + 1^2 - 2\cos 135°) - 1/\sqrt{2}$$

$$= \tfrac{1}{8}\pi(2 + \sqrt{2}) - 1/\sqrt{2}.$$

Multiplying by 4 gives $\pi(1 + 1/\sqrt{2}) - 2\sqrt{2} = \pi(2 + \sqrt{2})/2 - 2\sqrt{2}$.

Note: as an alternative to using the cosine rule to calculate OA, it is possible to apply Pythagoras' Theorem giving



$$OA^2 = \left(1 + \frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 = 2 + \sqrt{2}.$$

1.	A
2.	D
3.	C
4.	B
5.	C
6.	E
7.	D
8.	D
9.	D
10.	C
11.	B
12.	B
13.	D
14.	E
15.	B
16.	D
17.	A
18.	D
19.	B
20.	A
21.	C
22.	D
23.	E
24.	A
25.	E

UK SENIOR MATHEMATICAL CHALLENGE

Organised by the United Kingdom Mathematics Trust

SOLUTIONS

Keep these solutions secure until after the test on

FRIDAY 20 NOVEMBER 1998

This solutions pamphlet outlines a solution for each problem on this year's paper. We have tried to give the most straightforward approach, but the solutions presented here are not the only possible solutions. Occasionally we have added a 'Note' (in italics).

Please share these solutions with your students.

Much of the potential benefit of grappling with challenging mathematical problems depends on teachers making time for some kind of review, or follow-up, during which students may begin to see what they should have done, and how many problems they could have solved.

We hope that you and they agree that the first 15 problems could, in principle, have been solved by most candidates; if not, please let us know.

1. A The sum, 60, is 5 times the middle number. So the middle number is 12. The numbers are 8, 10, 12, 14, 16.
Note: consecutive even numbers form an arithmetic sequence and the middle term of such a sequence with an odd number of terms is always the mean of all the terms.

2. D Of the thirty pupils, 8 are left-handed. In the extreme case, these will all be girls. So the smallest number of girls who could be right-handed is 6.

3. C Multiplying the decimal part of each option by 8 gives 0.40, 1.20, 2, 2.8 and 3.6.

4. B The diameter of the smaller circle and the radius of the larger circle are equal. Letting the radius of the smaller circle be r, the areas are πr^2 and $\pi(2r)^2 = 4\pi r^2$ so the smaller circle is $\frac{1}{4}$ of the larger.
Note: this is an example of the property that in an enlargement the area scale factor is the square of the linear factor.

5. C The perimeter of the rectangle is 12 cm. So each side of the square is 3 cm.

6. E The question really asks how many of the first 15 positive integers can be expressed as the sum of 4 or fewer perfect squares. These can be listed as follows:–
$1 = 1^2$; $2 = 1^2 + 1^2$; $3 = 1^2 + 1^2 + 1^2$; $4 = 2^2$. Expressions for 5, 6, 7 and 8 can be obtained by adding 2^2 to each of the ones already given. $9 = 3^2$ and expressions for 10, 11, 12, 13, 14, and 15 can be obtained by adding 3^2 to the ones for 1, 2, 3, 4, 5 and 6.
Note: In fact, the French mathematician Lagrange proved in 1770 that every positive integer can be expressed as the sum of four squares.

7. D Let Mary's height at age 5 be h. So at age 10 her height is $1.3h$ and by age 15 this will be $1.2 \times 1.3h = 1.56h$.

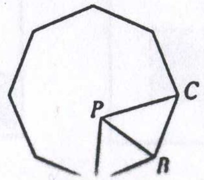
8. D Squaring each number gives 240, 250, 245, 243 and 242. In terms of size, the middle of these is 243.

9. D Using their initial to denote a candidate's mark gives:–
$P + N = 32$; $P + F = 26$; $N + F = 36$. Adding these three equations gives $2P + 2N + 2F = 94$, hence $P + N + F = 47$. Thus $47 + G = 4 \times 16$ giving $G = 17$.
Biographical note: although the lives of Pascal and Fermat overlapped with the lives of the other two mathematicians, Galileo died in 1642, the year in which Newton was born (on Christmas Day).

10. C There are 60×60 seconds in an hour. In a day there are $24 \times 60 \times 60 = 24 \times 4 \times 15 \times 60 = 96 \times 900$ which is roughly $10^2 \times 10^3 = 10^5$.

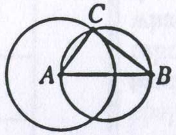
11. B $\frac{1647}{8235} = \frac{1}{5}$ becomes $\frac{167}{835} = \frac{1}{5}$ which in turn becomes $\frac{17}{85} = \frac{1}{5}$. Hence $D = 85$ and $C = 17$ so $D - C = 68$.

12. B ΔAPB is equilateral hence $\angle ABP = 60°$.
But $BC = AB = PB$ so ΔPBC is isosceles with $\angle PBC = 135° - 60° = 75°$.
$\therefore \angle BPC = \frac{1}{2}(180 - 75)°$ and $\angle APC = 52.5° + 60°$.



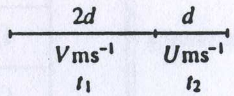
13. D
$$x = \left(\tfrac{1}{4}\right)^{\frac{1}{2}} = \sqrt{\tfrac{1}{4}} = \tfrac{1}{2}$$
$$\therefore x^{-x} = \left(\tfrac{1}{2}\right)^{-\frac{1}{2}} = (2^{-1})^{-\frac{1}{2}} = 2^{\frac{1}{2}} = \sqrt{2}.$$

14. E



AB is a diameter so $\angle ACB = 90°$.
$$\therefore CB^2 = AB^2 - AC^2 = 400 - 144 = 256$$
$$CB = 16$$

15. B



$$t_1 = \frac{2d}{V};\ t_2 = \frac{d}{U}$$
$$\therefore \frac{2d}{V} + \frac{d}{U} = T$$
$$d\left(\frac{2}{V} + \frac{1}{U}\right) = T$$
$$d(2U + V) = TUV$$
$$\therefore 3d = 3TUV/(2U + V)$$

16. D Number of years $= \frac{49}{6} \times \frac{48}{5} \times \frac{47}{4} \times \frac{46}{3} \times \frac{45}{2} \times \frac{44}{1} + 52$
$= 49 \times 1 \times 47 \times 46 \times 3 \times 44 + 52$
$\approx 50^3 \times 3 = 375\,000$.
Note: 375000 is clearly an overestimate since $49 \times 1 \times 47 \times 46$ *is less than* 50^3 *and* $3 \times 44 + 52$ *is less than 3.*

17. A Re-ordering $(x - 1)(x^4 + 1)(x^2 + 1)(x + 1) = (x - 1)(x + 1)(x^2 + 1)(x^4 + 1)$
$= (x^2 - 1)(x^2 + 1)(x^4 + 1) = (x^4 - 1)(x^4 + 1) = x^8 - 1$
Note: $(x - 1)(x^4 + 1)(x^2 + 1)(x + 1)$ *has value 0 when* $x = 1$. *This enables B, C and D to be eliminated.*

18. D Curved surface area of small cylinder $= 2\pi r \times 4r = 8\pi r^2$.
Curved surface area of large cylinder $= 4\pi r \times 4r = 16\pi r^2$.
Area of each end $= \pi(2r)^2 - \pi r^2 = 3\pi r^2$.
Total surface area $= 8\pi r^2 + 16\pi r^2 + 2 \times 3\pi r^2 = 30\pi r^2$.

19. B $\cos\theta = \frac{1}{2}$; $\sin^2\theta = \frac{3}{4}$; $\sin\theta = \pm\frac{1}{2}\sqrt{3}$
$\sin 2\theta = 2\cos\theta\sin\theta = 2 \times \frac{1}{2} \times (\pm\frac{1}{2}\sqrt{3}) = \pm\frac{1}{2}\sqrt{3}$
Note: $\sin 2\theta$ *is equal to* $2\cos\theta\sin\theta$ *for all values of* θ *and since in this case* $\cos\theta = \frac{1}{2}$, $\sin 2\theta$ *will also equal* $\sin\theta$.

20. A Space diagonal of the cube is 1 metre.
So, using x as the side of the cube
$$3x^2 = 1$$
$$x^2 = \frac{1}{3} = \text{area of one face}$$
$\therefore$ total area of all six faces is 2 m².