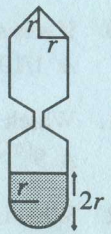
19. One end of an egg-timer is a hemisphere of radius r; the other end is a cone of radius r and height r. Both ends are attached to cylinders of radius r. When the hemisphere is at the bottom, the sand in the egg-timer comes to a height $2r$ above the lowest point. What is the corresponding height of the sand when the egg-timer has been turned over and all the sand has been allowed to run through to the other end?



A $\frac{4}{3}r$ *B* $\frac{5}{3}r$ *C* $2r$ *D* $\left(\frac{1}{2}+\frac{\pi}{4}\right)r$ *E* $\frac{7}{3}r$

20. In a triangle the perpendicular from a vertex to the opposite side is called an *altitude*. If h, h', h'' denote the lengths of the three altitudes of a triangle, which of the following ratios never occurs as the ratio $h : h' : h''$?

A 2 : 3 : 4 *B* 2 : 3 : 5 *C* 2 : 4 : 5 *D* 3 : 4 : 5 *E* 3 : 4 : 6

21. A square is divided into nine rectangles by two horizontal and two vertical lines. The areas of three of the small rectangles are as shown.

If the central small rectangle happens to be a square, what is the perimeter of the small rectangle in the bottom left corner?

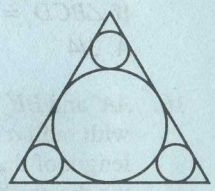
1		3
		6

A $2/\sqrt{3}$ *B* 2 *C* $3\sqrt{3}$ *D* 6 *E* $11/\sqrt{3}$

22. Which of the following expressions is identically equal to $\sin^3 x + \cos^3 x$?

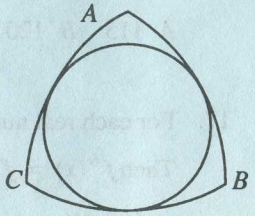
A $\sin 3x + \cos 3x$ *B* 1 *C* $(\sin x + \cos x)(1 - \sin x \cos x)$
D $(\sin x + \cos x)^3$ *E* $(\sin x + \cos x)(2 \sin x \cos x + 1)$

23. A circle is inscribed in an equilateral triangle. Small circles are then inscribed in each corner as shown. What is the ratio of the area of a small circle to that of the large circle?



A 1 : 3 *B* 1 : 4.5 *C* 1 : $3\sqrt{3}$ *D* 1 : 6 *E* 1 : 9

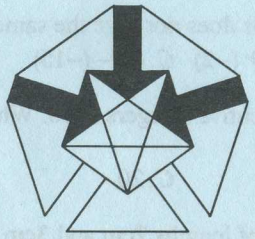
24. The curvy shape *ABC* shown here is called a *Reuleaux* triangle (after the French engineer *Franz Reuleaux* (1829-1905)). Its perimeter consists of three equal arcs *AB*, *BC*, *CA* each with the same radius and centred at the opposite vertex. In the *Reuleaux* triangle shown, each arc has radius 3cm. What is the area (in cm^2) of the inscribed circle?



A $6\pi(2-\sqrt{3})$ *B* $9\pi/4$ *C* $2\pi(3-\sqrt{3})$ *D* $3\pi/4$ *E* 9π

25. A right circular cone has apex angle 2α. A sphere is inscribed in the cone, touching the base. What fraction of the cone is occupied by the sphere?

A $2\sin\alpha\cos 2\alpha/(1+\cos\alpha)^3$ *B* $4\sin\alpha(1-\sin\alpha)/\cos^2\alpha$ *C* $4(1-\sin\alpha)^3\sin\alpha/\cos^4\alpha$
D $\sin 2\alpha\cos\alpha/(1+\sin\alpha)^3$ *E* need more information



UK SENIOR MATHEMATICAL CHALLENGE

FRIDAY 21 NOVEMBER 1997

Organised by the **United Kingdom Mathematics Trust**

RULES AND GUIDELINES (to be read before starting)

1. Do not open the question paper until the invigilator tells you to do so.
2. Detach the Answer Sheet (back page) and fill in your personal details before you open the question paper and begin.
   Once you have begun, record all your answers on the Answer Sheet.
3. Time allowed: **90 minutes.**
   No answers or personal details may be entered on the Answer Sheet after the 90 minutes are over.
4. The use of rough paper is allowed.
   Calculators, measuring instruments and squared paper are forbidden.
5. Candidates must be full-time students at secondary school or FE college, and must be in Year 13 or below (England & Wales); S6 or below (Scotland); Year 14 or below (Northern Ireland).
6. There are twenty-five questions. Each question is followed by five options marked *A*, *B*, *C*, *D*, *E*. Only one of these is correct. Enter the letter *A*-*E* corresponding to the correct answer in the corresponding box on the Answer Sheet.
7. **Scoring rules**: all candidates start out with 25 marks;
   0 marks are awarded for each question left unanswered;
   4 marks are awarded for each correct answer;
   1 mark is deducted for each incorrect answer.
8. **Guessing**: Remember that there is a penalty for wrong answers. Note also that later questions are deliberately intended to be harder than earlier questions. You are thus advised to concentrate first on solving as many as possible of the first 15-20 questions. Only then should you try later questions.

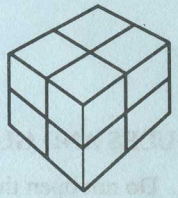
1. Which of these calculations does not give the same answer as the rest?
   A $2+4\times2$ *B* $(-24)\div(-2)$ *C* $-3-(-15)$ *D* $24\times(0.5)$ *E* $(-7)+19$

2. The average of five consecutive integers is 10. What is the sum of the second and fourth of these integers?
   A 6 *B* 10 *C* 20 *D* 21 *E* can't be sure

3. A rhombus has diagonals of lengths 2cm and 3cm. What is its area (in cm^2)?
   A 2 *B* 3 *C* 4 *D* 5 *E* 6

4. Timmy Riddle was selling toffee apples at the school fête. When I asked him what they cost he said "One toffee apple costs the smallest amount that cannot be paid exactly using four or fewer standard British coins".
   I bought as many toffee apples as I could get for £1. How much change did I receive?
   A 1p *B* 5p *C* 18p *D* 22p *E* 24p

5. Eight unit cubes are arranged to form an imaginary 2 by 2 by 2 cube. What is the largest number of unit cubes one can remove from this arrangement if the resulting shape has to have the same surface area as the original?



   A 0 *B* 1 *C* 2 *D* 3 *E* 4

6. Here are the measurements of five different rectangles. The list includes two pairs of *similar* rectangles, while one rectangle differs in shape from all the others. Which is the odd one out?
   A 240×120 *B* 300×180 *C* 55×110 *D* 320×200 *E* 210×350

7. The first six volumes of the *Encyclopedia of Mathematicians* are arranged in order on my shelf from left to right. The six volumes contain names beginning A-Ba, Be-Ca, Ce-Ei, Ek-Fe, Fee-Fi, Fo-Fum. If one ignores the covers, which of the following encyclopedia entries could be on a page "next to" the page with the entry for Einstein?
   A Abel *B* Bernoulli *C* Cantor *D* Euler *E* Fibonacci

8. A teacher gave a test to 20 students. Marks on the test ranged from 0 to 10 inclusive. The average of the first twelve papers marked was 6.5. What can you conclude from this about the eventual average M for the whole group?
   A $0.325\leqslant M\leqslant6.5$ *B* $3.25\leqslant M\leqslant6.5$ *C* $3.9\leqslant M\leqslant6.5$
   D $3.9\leqslant M\leqslant7.9$ *E* $6.5\leqslant M\leqslant7.9$

9. A cube $ABCDEFGH$ has $ABCD$ as square base, with E, F, G, H above A, B, C, D respectively. What is the cosine of the angle $\angle CAG$?
   A $1/\sqrt{3}$ *B* $\sqrt{2}/3$ *C* $1/\sqrt{2}$ *D* $\sqrt{(2/3)}$ *E* $\sqrt{3}/2$

10. If n is some integer, $1\leqslant n\leqslant9$, what is the value of $(0.n)/(0.\dot{n})$?
    A 1/10 *B* 9/10 *C* 1 *D* 10/9 *E* it depends on n

11. Which of these five expressions represents the largest number?
    A $9^{(9^9)}$ *B* 999 *C* 9^{99} *D* $(9^9)^9$ *E* 99^9

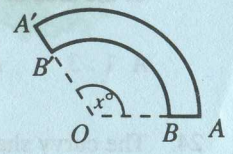
12. What is the value of
    $1+2(1+2(1+2(1+2(1+2(1+2(1+2(1+2(1+2(1+2(1+2)))))))))?$
    A $2^{10}+1$ *B* $2^{11}-1$ *C* $2^{11}+1$ *D* $2^{12}-1$ *E* $2^{12}+1$

13. Last year Noel bought a number of identically priced Christmas cards. The total cost was £15.60. In a gesture of seasonal goodwill the shopkeeper gave him one extra card free, and this reduced the average cost per card by exactly 1p. At their original price how many cards could Noel have bought with £5?
    A 8 *B* 12 *C* 16 *D* 20 *E* 24

14. I am trying to do a rectangular jigsaw puzzle. The puzzle was made by starting with a rectangular picture and then cutting it into 1000 pieces by sawing along the lines of a (wiggly!) rectangular grid. I start by separating out all the edge and corner pieces. Which of the following could *not* possibly be the number of corner and edge pieces of such a jigsaw?
    A 126 *B* 136 *C* 216 *D* 316 *E* *A-D* are all possible

15. Triangle ABC is isosceles with $AB = AC$, and D is the midpoint of AB.
    If $\angle BCD = \angle BAC = \theta$, then $\cos\theta$ equals
    A 3/4 *B* $\sqrt{7}/(2\sqrt{2})$ *C* $1/\sqrt{2}$ *D* $\sqrt{7}/4$ *E* $1/(2\sqrt{2})$

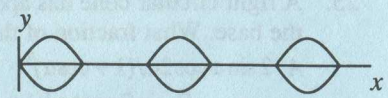
16. AA' and BB' are arcs of concentric circles with centre O and with radii a and b respectively. Let $\angle A'OA = x°$. The length of the arc AA' is equal to the total distance from A to A' via the arc BB'. Find the value of x to the nearest integer.



    A 115 *B* 120 *C* 125 *D* 135 *E* it depends on a and b

17. For each real number x, except $x = 0, 1, -1$, let $f(x) = \dfrac{(x-1)}{(x+1)}$.
    Then $f^6(x) = f(f(f(f(f(f(x))))))$ equals
    A $\dfrac{(x+1)}{(x-1)}$ *B* $\dfrac{-1}{x}$ *C* x *D* $\dfrac{1}{x}$ *E* $\left[\dfrac{x-1}{x+1}\right]^6$

18. Which of the following equations could be the equation of the "curve", part of which is shown on the right?



    A $y=\sin x$ *B* $|y|=\sin x$ *C* $y=|\sin x|$ *D* $|y|=|\sin x|$ *E* *A-D* all impossible