20. **C** The five options all have $h < h' < h''$. Label the triangle ABC so that h, h', h'' are the altitudes from A, B, C respectively. ΔABC has area $\Delta = ah/2 = bh'/2 = ch''/2$.

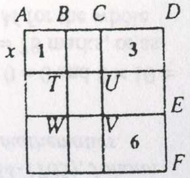
$$\therefore a > b > c \text{ (since } h < h' < h''\text{)}.$$

Three positive lengths a, b, c with $a > b > c$ form the sides of a triangle precisely when $a < b + c$ (by the triangle inequality). This condition is equivalent to $a/2\Delta < b/2\Delta + c/2\Delta$; that is,

$$1/h < 1/h' + 1/h''.$$

The ratios A, B, D, E all satisfy this condition, but C does not.

21. **E** Suppose the large square has side s, and the small top left rectangle has vertical side of length x.



$\therefore$ $AB = 1/x$, $CD = 3/x$, $EF = 2x$.

$\therefore$ $TU = s - 4/x$, $TW = s - 3x$.

$\therefore$ $s - 4/x = 2 - 3x$ (since $TUVW$ is a square – given)

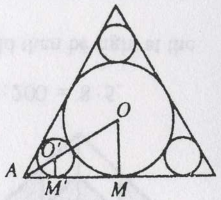
$\therefore$ $3x = 4/x$, so $x = 2/\sqrt{3}$.

$\therefore$ perimeter of the bottom left rectangle $= 2/x + 4x = 3/\sqrt{3} + 8/\sqrt{3} = 11/\sqrt{3}$.

22. **C** $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

$\therefore$ $\sin^3 x + \cos^3 x = (\sin x + \cos x)(\sin^2 x - \sin x \cos x + \cos^2 x) = (\sin x + \cos x)(1 - \sin x \cos x)$

23. **E** ΔAOM is a 30-60-90 triangle, so $OA = 2OM$.



ΔAOM and $\Delta AO'M'$ are similar, and

$OO' = OM + O'M'$ (sum of two radii);

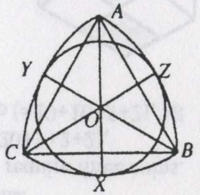
$$\therefore \frac{O'M'}{OM} = \frac{O'A}{OA} = \frac{OA - OO'}{OA} = 1 - \frac{OO'}{OA}$$

$$= 1 - \frac{(OM + O'M')}{2.OM} = \frac{1}{2} - \frac{OM'}{2.OM}$$

$$\therefore O'M' : OM = 1 : 3$$

$$\therefore \pi(O'M')^2 : \pi(OM)^2 = 1 : 9$$

24. **A** Join AB, BC and CA.



Then ΔABC is equilateral, with sides length 3cm.

The three medians meet at O, where

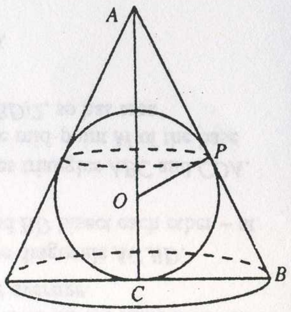
$AO = BO = CO = (2/3) \times (3\sqrt{3}/2) = \sqrt{3}$.

$\therefore$ $OX = OY = OZ = 3 - \sqrt{3}$.

Hence the circle centre O through X, Z, Z has area

$\pi(3 - \sqrt{3})^2 = \pi(12 - 6\sqrt{3}) = 6\pi(2 - \sqrt{3})$.

25. **C** Let A be the apex of the cone, O the centre of the sphere, and C the point where the sphere touches the base of the cone. Let P be any point where the sphere touches the side of the cone, and let B be the point where the line AP meets the base of the cone.



Let $OP = r = OC$ be the radius of the sphere.

ΔABC and ΔAOP are similar.

$\therefore$ $BC/OP = AC/AP = (AO + OC)/AP$

$\therefore$ $BC/r = [r/\sin\alpha + r]/[r/\tan\alpha]$

$\therefore$ $BC = r(1 + \sin\alpha)/\cos\alpha$

$\therefore$ volume of sphere $= \mathcal{S} = 4\pi r^3/3$

volume of cone $= \mathcal{C} = (1/3)\pi[r(1 + \sin\alpha)/\cos\alpha]^2 \times [r/\sin\alpha + r]$

$$\therefore \quad \mathcal{S}/\mathcal{C} = 4\left[\left(\frac{1 + \sin\alpha}{\cos\alpha}\right)^2\left(\frac{1 + \sin\alpha}{\sin\alpha}\right)\right]^{-1} = 4\sin\alpha\cos^2\alpha/(1 + \sin\alpha)^3$$

$$= \frac{4\sin\alpha\cos^2\alpha(1 - \sin\alpha)^3}{(1 - \sin^2\alpha)^3} = \frac{4\sin\alpha(1 - \sin\alpha)^3}{\cos^4\alpha}.$$

UK SENIOR MATHEMATICAL CHALLENGE

Organised by the United Kingdom Mathematics Trust

SOLUTIONS

Keep these solutions secure until after the test on

FRIDAY 21 NOVEMBER 1997

This solutions pamphlet outlines a solution for each problem on this year's paper. We have tried to give the most straightforward approach, but the solutions presented here are not the only possible solutions. Occasionally we have added a 'Note' (in italics).

Please share these solutions with your students.

Much of the potential benefit of grappling with challenging mathematical problems depends on teachers making time for some kind of review, or follow-up, during which students may begin to see what they should have done, and how many problems they could have solved.

We hope that you and they agree that the first 15 problems could, in principle, have been solved by most candidates; if not, please let us know.

1.	A
2.	C
3.	B
4.	E
5.	E
6.	D
7.	B
8.	D
9.	D
10.	B
11.	A
12.	D
13.	B
14.	D
15.	A
16.	A
17.	B
18.	B
19.	E
20.	C
21.	E
22.	C
23.	E
24.	A
25.	C

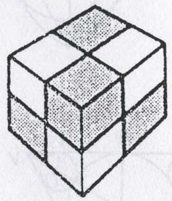
1. A $2 \times 4 + 2 = (2 \times 4) + 2 = 10$; the other options all give the answer 12.

2. C The five integers are "consecutive".
The third term is the middle term and so is equal to the average 10, while the second and fourth terms are equal to $10 - d$ and $10 + d$, where $d = 1$ is the common difference.
NOTE: (a) There is no need to work out the terms of the sequence.
(b) The same idea will work for any AP: the sum of two terms the same distance either side of the middle term is equal to twice the average.

3. B The area of a rhombus $ABCD$ is equal to half the product of the diagonals $AC.BD$.
[A rhombus is a special parallelogram, so the diagonals AC and BD bisect each other – at M say.
The diagonal AC cuts the rhombus into two congruent isosceles triangles ABC and CDA.
The line joining the apex B of the isosceles triangle ABC to the mid-point M of the base AC is perpendicular to AC; $\therefore$ ΔABC has base AC and height $BD/2$, so has area $(AC/2).(BD/2)$. $\therefore$ the rhombus $ABCD$ has area $(AC.BD)/2$.]

4. E The smallest amount that cannot be paid with $\leqslant 4$ coins is 38p.
[1p, 2p, 5p, 10p, 20p can all be paid with just one coin.
3p, 4p, 6p, 7p, 11p, 12p, 15p, 21p, 22p, 25p, 30p require two coins.
8p, 9p, 13p, 14p, 16p, 17p, 23p, 24p, 26p, 27p, 31p, 32p, 35p all require three coins.
18p (=10+5+2+1), 19p (=10+5+2+2), 28p (=20+5+2+1), 29p (=20+5+2+2),
33p (=20+10+2+1), 34p (=20+10+2+2), 36p (=20+10+5+1), 37p (=20+10+5+2), all require four coins. 38p cannot be paid with four coins.]

5. E The surface of the initial 2 by 2 by 2 arrangement consists of $6 \times 2^2 = 24$ unit squares. The surface of each unit cube consists of just 6 unit squares. Hence we have to leave at least 4 unit cubes to ensure a surface area of 24 unit squares.
In fact, if one removes the unit cubes, with shaded faces, at the four corners, the resulting "shape" has a surface area of exactly 24 unit squares.



6. D $240:120 = 2:1 = 110:55$; $300:180 = 5:3 = 350:210$. But $320:200 = 8:5$.

7. B The entry for *Bernoulli* could be the first entry in Volume 2; it would then be right at the front – that is, on the *right* of Volume 2 as it stands on the shelf.
The entry for *Einstein* could be the last entry in Volume 3, so would be right at the back - on the *left* of Volume 3 as it stands on the shelf.
NOTE: There would be several entries under the name of "Bernoulli"!
In fact the Professor of Mathematics in the University of Basel was called Bernoulli for 105 years in a row. Three members of the family stand out: Jakob (1654-1705), Johann (1667-1748) and Daniel (1700-1782) all made major contributions to mathematics.

8. D The first twelve students scored a total of $12 \times 6.5 = 78$ marks.
The remaining eight students might have scored any total between $8 \times 0 = 0$ and $8 \times 10 = 80$. Thus the twenty students could have a total score as low as $78 + 0 = 78$ marks, or as high as $78 + 80 = 158$ marks. Hence all we can say about the average M for the whole group is that it must be $\geqslant 78/20 = 3.9$ and $\leqslant 158/20 = 7.9$.

9. D Let the cube have side length s. Using Pythagoras on the right angled triangle ABC (with $AB = BC = s$), we find $AC = s\sqrt{2}$.
Using Pythagoras on the right angled triangle ACG, we find $AG = s\sqrt{3}$.
Hence $\cos\angle CAG = AC/AG = \sqrt{(2/3)}$.

10. B $0.1 = 1/10$, so $0.n = n/10$. $0.\dot{1} = 1/9$, so $0.\dot{n} = n/9$.

11. A $999 = 9 \times 111 < 9 \times 9^3 = 9^4 < 99^9 < (9^3)^9 = 9^{27} < 9^{81} = (9^9)^9 < 9^{99} < 9^{(9^9)}$

12. D $1 + 2(1 + 2(1 + 2(1 + 2(1 + 2(1 + 2(1 + 2(1 + 2(1 + 2(1 + 2(1 + 2))))))))))$
$= 1 + 2 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6 + 2^7 + 2^8 + 2^9 + 2^{10} + 2^{11}$.
Now $(x^{n-1} + x^{n-2} + \ldots + x^2 + x + 1)(x - 1) = (x^n - 1)$
$\therefore 2^{11} + 2^{10} + 2^9 + \ldots + 2^2 + 2 + 1) = (2^{12} - 1)/(2 - 1) = 2^{12} - 1$.

13. B Suppose Noel bought c cards at n pence each. Thus $n.c = 1560$. The extra (free) card effectively reduced the unit cost by 1p.
$\therefore \quad (n + 1)(c - 1) = 1560$
$\therefore \quad nc + c - n - 1 = 1560$
$\therefore \quad c = n + 1$ (since $nc = 1560$)
$\therefore \quad n.c = n(n + 1) = 1560$
$\therefore \quad 0 = n^2 + n - 1560 = (n - 39)(n + 40)$
$\therefore \quad n = 39$ (since $n > 0$).
Hence Noel could have bought 12 cards for £5 (with 32p change).

14. D Suppose there are h pieces along each horizontal edge and v pieces along each vertical edge. Then $h.v = 1000 = 2^3 5^3$
Thus the only possibilities for the pair $\{h, v\}$ are $\{1,1000\}$, $\{2, 500\}$, $\{4, 250\}$, $\{5, 200\}$, $\{8, 125\}$, $\{10, 100\}$, $\{20, 50\}$, $\{25, 40\}$.
The total number of edge pieces is $2h + 2v - 4 = 2(h + v - 2)$ (since $2h + 2v$ counts each of the four corners twice).
$\therefore$ $126 (= 2(25 + 40 - 2))$, $136 (= 2(20 + 50 - 2))$, $216 (= 2(10 + 100 - 2))$ are all possible; but 316 is not.

15. A The exterior angle at D of triangle ADC is equal to the sum of the two interior opposite angles:
$\therefore \angle BDC = \angle DAC + \angle DCA$
$= \theta + (\angle ACB - \angle BCD) = \theta + (\angle ACB - \theta) = \angle ACB = \angle DBC$ (given).
Hence ΔCBD is isosceles with the same angles as ΔACB.
$\therefore AC/BC = BC/BD = BC/(AC/2)$
$\therefore \quad AC^2 = 2.BC^2$
$\therefore \quad BC^2 = AC^2 + AB^2 - 2.AC.AB.\cos\theta$ (by the cosine rule in ΔABC)
$\therefore \quad BC^2 = (\sqrt{2}.BC)^2 + (\sqrt{2}.BC)^2 - 2.(\sqrt{2}.BC)(\sqrt{2}.BC)\cos\theta$
$\therefore \quad \cos\theta = (3.BC^2)/(4.BC^2) = 3/4$.

16. A Each arc is $x/360$ times the full circumference.
$\therefore \quad \text{arc}\,AA' = (x/360)2\pi a$, and $\text{arc}\,BB' = (x/360)2\pi b$.
Now equate the lengths of the two routes (arc AA' direct and via arc BB').
$\therefore \quad (x/360)2\pi a = (a - b) + (x/360)2\pi b + (a - b)$
$\therefore \quad (x/360)2\pi(a - b) = 2(a - b)$
$\therefore \quad (x/360)\pi = 1$
$\therefore \quad x = 360/\pi \approx 360/(22/7) = 1260/11 \approx 114.5$.

17. B $f^2(x) = f(f(x)) = f\left(\frac{x-1}{x+1}\right) = \left(\frac{x-1}{x+1} - 1\right)/\left(\frac{x-1}{x+1} + 1\right) = -2/2x = -1/x$.
$\therefore f^4(x) = f^2(f^2(x)) = f^2(-1/x) = -1/[-1/x] = x$.
$\therefore f^6(x) = f^2(f^4(x)) = f^2(x) = -1/x$.

18. B If (x, y) lies on the curve, so does $(x, -y)$: this excludes A, C.
Values of x which produce negative values of $\sin x$ do not feature: this excludes D. B is possible: this excludes E.

19. E Vol(sand) = vol(cylinder) + vol(hemisphere) = $(\pi r^2).r + \frac{2}{3}\pi r^3 = \frac{5}{3}\pi r^3$.
When turned over, the conical end (volume $(\pi r^2/3)r$) fills up first, and the cylinder then fills to a height of $4r/3$ – a total height $r + 4r/3 = 7r/3$.