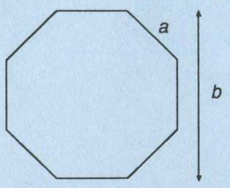
21. C



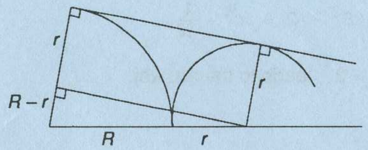
$$b = a + 2 \times \frac{a}{\sqrt{2}} = a(1+\sqrt{2}).$$

Vol of solid $= b^3 - 8$ (vol of corner tetrahedron)

$$= b^3 - 8 \times \frac{1}{3}\left(\frac{a}{\sqrt{2}}\right)^3.$$

22. B A congruent curve through (0,0) will have the equation $y = kx^2(x^2 - a)$ where $k > 0$ and $a > 0$. The equation $y = x^4 - 2x^2 = x^2(x^2-2)$ has $k = 1$ and $a = 2$, so $y = x^4 - 2x^2 - 3$ is a possible equation for the given graph. Similarly, $y = 3x^4 - 2x^2 - 1$ is a possible equation.

23. B



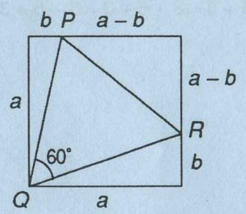
$$\frac{R-r}{R+r} = \sin 30^\circ \Rightarrow 2R - 2r = R + r$$

$$\Rightarrow r = \frac{1}{3}R.$$

(So radii and areas form G.P.'s.)

The areas of the circles are πR^2, $\frac{1}{9}\pi R^2$, $\left(\frac{1}{9}\right)^2\pi R^2$, ... Total area $= \pi R^2 \sum_{1}^{\infty}\left(\frac{1}{9}\right)^{n-1}$.

24. B



$PQ = PR$

$\Rightarrow a^2 + b^2 = 2(a-b)^2 = 2a^2 - 4ab + 2b^2$,

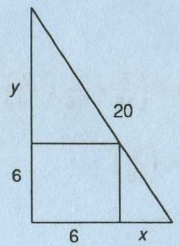
$0 = a^2 - 4ab + b^2$

$$\Rightarrow b = \frac{4a \pm\sqrt{16a^2 - 4a^2}}{2} = a(2 \pm \sqrt{3}),$$

$b < a \Rightarrow b = a(2 - \sqrt{3})$.

The area of the equilateral triangle $= \frac{1}{2} \times PQ^2 \sin 60^\circ = (a-b)^2 \sin 60^\circ$.

25. A



$(x+6)^2 + (y+6)^2 = 20^2$ and $\frac{6}{x} = \frac{y}{6}$

$\Rightarrow x^2 + y^2 + 12x + 12y + 72 = 400$ and $xy = 36$

$\Rightarrow x^2 + 2xy + y^2 + 12(x+y) - 400 = 0$

$\Rightarrow (x+y)^2 + 12(x+y) - 400 = 0$

$\Rightarrow x + y = \frac{1}{2}(-12 + \sqrt{144 + 1600}) = -6 + 2\sqrt{109}$.

Perimeter $= 32 + x + y = 26 + 2\sqrt{109}$.

UNITED KINGDOM SENIOR MATHEMATICAL CHALLENGE

FRIDAY 15th NOVEMBER 1996

SOLUTIONS

This solutions pamphlet outlines a solution for each problem on this year's contest. The solutions are not the only possible solutions, though we have tried to give the most straightforward approach.

Please share these solutions with your students. Much of the benefit they could gain from grappling with challenging mathematical problems gets lost without some kind of review, or follow-up, during which students realise what they should have done, and how many problems they *could have solved.*

We hope that you and they agree that the first 15 or more problems could, in principle, have been solved by most participants: if not, please let us know.

B	1	14	D
D	2	15	B
A	3	16	A
D	4	17	C
E	5	18	E
B	6	19	A
B	7	20	E
C	8	21	C
A	9	22	B
D	10	23	B
D	11	24	B
B	12	25	A
E	13		

ANSWER KEY
Scoring

Correct 4

Blank 0

Wrong −1

1. **B** Apply the rules for the multiplication and addition of odd and even numbers.

2. **D** $4 \times 3 + 2 + 1$ = number of blocks.

3. **A** $2^2, 4^2, 6^2, \ldots, 30^2$ are the possible square numbers.

4. **D** Apply the rules for the order of the operations multiplication, addition and subtraction.

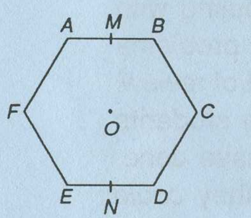
5. **E** Gradient = $\dfrac{\text{increase in } y}{\text{corresponding increase in } x}$

6. **B** By proportion, 2×10^{11} cells are produced in 24 hours

$\Rightarrow$ 1.25×10^{10} cells are produced in 90 minutes.

7. **B** The next square up is $(\sqrt{n}+1)^2$.

8. **C**



$MN = 2MO = 2\sqrt{2^2-1^2} = 2\sqrt{3}$,

$EM = \sqrt{1^2+12} = \sqrt{13}$.

9. **A** Profit 10% $\Rightarrow$ cost price = £$\dfrac{9999}{1.1}$ = £$\dfrac{99990}{11}$, that is a profit of £909;

loss 10% $\Rightarrow$ cost price = £$\dfrac{9999}{0.9}$ = £$\dfrac{99990}{9}$, that is a loss of £1111.

10. **D** If v_0 = original value and v_1 = new value, then

$\dfrac{1}{v_0} = \dfrac{1}{f} - \dfrac{1}{u}$ and $\dfrac{1}{v_1} = \dfrac{2}{f} - \dfrac{2}{u}$ $\Rightarrow$ $\dfrac{1}{v_1} = \dfrac{2}{v_0}$.

11. **D** $AP = PH = HQ = AQ$ as they are the hypotenuses of the congruent triangles APB, HPC, HQE, AQF, respectively;

$AP \parallel QM$; $\angle QAP \neq 90^\circ$.

12. **B** Linear scale factor = 1.03 $\Rightarrow$ area scale factor = 1.03^2 = 1.0609.

13. **E** $xy = 2(x+y) \Rightarrow \dfrac{1}{2} = \dfrac{x+y}{xy} = \dfrac{1}{y} + \dfrac{1}{x}$.

14. **D** Section 1 distances increases more and more quickly.
Section 2 distances increase steadily.
Section 3 distances increase more and more slowly.

15. **B** $968880726456484032 \times 875 = 968880726456484032 \times 1000 \div 8 \times 7$.

16. **A** $EF = FC \Rightarrow \angle ECF = x^\circ \Rightarrow \angle CFA = 2x^\circ$,

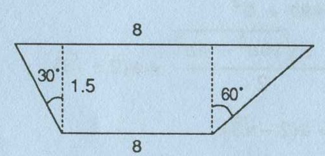
$FABC$ is a cyclic quadrilateral $\Rightarrow \angle ABC = 180^\circ - 2x^\circ$.

17. **C** $8^2 = \left(x + \dfrac{1}{x}\right)^2 = x^2 + 2 + \dfrac{1}{x^2} \Rightarrow 8^2 - 2 = x^2 + \dfrac{1}{x^2}$.

Squaring again, and remembering that $8 = 2^3$, leads to the answer.

18. **E** $9+8+7+6+5+4+3+2+1+0 = 45$ and $198 - 45 = 153$.
Omitting a '+' increases the sum by 9 times the preceding digit
(for example, $4 + 3 = 7$, but $43 = 4 \times 10 + 3 = 4 \times 9 + 7$).
$153 \div 9 = 17$ so consider the partitions of 17:
$17 = 9+7+1$ (the given example) $= 9+6+2 = 9+5+3 = 8+6+3 = 8+5+3+1$,
these work but the other partitions of 17 fail.

19. **A**



Distance boat travels = $(1.5\tan 30^\circ + 8 + 1.5\tan 60^\circ)$ km.

Time = 2 hours. Average speed = $\dfrac{\text{distance}}{\text{time}}$.

20. **E** Consider the relevant partitions of six: Probability $(1+2+3) = 6\left(\dfrac{6}{16} \times \dfrac{8}{16} \times \dfrac{2}{16}\right)$,

Probability $(2+2+2) = \left(\dfrac{8}{16}\right)^2$.

Total probability = $\dfrac{9}{64} + \dfrac{1}{8} = \dfrac{17}{64}$.