## OXFORD CAMBRIDGE AND RSA EXAMINATIONS

# Advanced Subsidiary General Certificate of Education Advanced General Certificate of Education 

MEI STRUCTURED MATHEMATICS

4768

Statistics 3
Thursday 12 JANUARY 2006 Afternoon 1 hour 30 minutes

Additional materials:
8 page answer booklet
Graph paper
MEI Examination Formulae and Tables (MF2)

TIME 1 hour 30 minutes

## INSTRUCTIONS TO CANDIDATES

- Write your name, centre number and candidate number in the spaces provided on the answer booklet.
- Answer all the questions.
- You are permitted to use a graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.


## INFORMATION FOR CANDIDATES

- The number of marks is given in brackets [ ] at the end of each question or part question.
- You are advised that an answer may receive no marks unless you show sufficient detail of the working to indicate that a correct method is being used.
- The total number of marks for this paper is 72 .

1 A railway company is investigating operations at a junction where delays often occur. Delays (in minutes) are modelled by the random variable $T$ with the following cumulative distribution function.

$$
\mathrm{F}(t)= \begin{cases}0 & t \leqslant 0 \\ 1-\mathrm{e}^{-\frac{1}{3} t} & t>0\end{cases}
$$

(i) Find the median delay and the 90th percentile delay.
(ii) Derive the probability density function of $T$. Hence use calculus to find the mean delay. [5]
(iii) Find the probability that a delay lasts longer than the mean delay.

You are given that the variance of $T$ is 9 .
(iv) Let $\bar{T}$ denote the mean of a random sample of 30 delays. Write down an approximation to the distribution of $\bar{T}$.
(v) A random sample of 30 delays is found to have mean 4.2 minutes. Does this cast any doubt on the modelling?

2 Geoffrey is a university lecturer. He has to prepare five questions for an examination. He knows by experience that it takes about 3 hours to prepare a question, and he models the time (in minutes) taken to prepare one by the Normally distributed random variable $X$ with mean 180 and standard deviation 12 , independently for all questions.
(i) One morning, Geoffrey has a gap of 2 hours 50 minutes ( 170 minutes) between other activities. Find the probability that he can prepare a question in this time.
(ii) One weekend, Geoffrey can devote 14 hours to preparing the complete examination paper. Find the probability that he can prepare all five questions in this time.

A colleague, Helen, has to check the questions.
(iii) She models the time (in minutes) to check a question by the Normally distributed random variable $Y$ with mean 50 and standard deviation 6 , independently for all questions and independently of $X$. Find the probability that the total time for Geoffrey to prepare a question and Helen to check it exceeds 4 hours.
(iv) When working under pressure of deadlines, Helen models the time to check a question in a different way. She uses the Normally distributed random variable $\frac{1}{4} X$, where $X$ is as above. Find the length of time, as given by this model, which Helen needs to ensure that, with probability 0.9 , she has time to check a question.

Ian, an educational researcher, suggests that a better model for the time taken to prepare a question would be a constant $k$ representing "thinking time" plus a random variable $T$ representing the time required to write the question itself, independently for all questions.
(v) Taking $k$ as 45 and $T$ as Normally distributed with mean 120 and standard deviation 10 (all units are minutes), find the probability according to Ian's model that a question can be prepared in less than 2 hours 30 minutes.

Juliet, an administrator, proposes that the examination should be reduced in time and shorter questions should be used.
(vi) Juliet suggests that Ian's model should be used for the time taken to prepare such shorter questions but with $k=30$ and $T$ replaced by $\frac{3}{5} T$. Find the probability as given by this model that a question can be prepared in less than $1 \frac{3}{4}$ hours.

3 A production line has two machines, A and B, for delivering liquid soap into bottles. Each machine is set to deliver a nominal amount of 250 ml , but it is not expected that they will work to a high level of accuracy. In particular, it is known that the ambient temperature affects the rate of flow of the liquid and leads to variation in the amounts delivered.

The operators think that machine B tends to deliver a somewhat greater amount than machine A, no matter what the ambient temperature. This is being investigated by an experiment. A random sample of 10 results from the experiment is shown below. Each column of data is for a different ambient temperature.

| Ambient <br> temperature | $T_{1}$ | $T_{2}$ | $T_{3}$ | $T_{4}$ | $T_{5}$ | $T_{6}$ | $T_{7}$ | $T_{8}$ | $T_{9}$ | $T_{10}$ |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Amount <br> delivered by <br> machine A | 246.2 | 251.6 | 252.0 | 246.6 | 258.4 | 251.0 | 247.5 | 247.1 | 248.1 | 253.4 |
| Amount <br> delivered by <br> machine B | 248.3 | 252.6 | 252.8 | 247.2 | 258.8 | 250.0 | 247.2 | 247.9 | 249.0 | 254.5 |

(a) Use an appropriate $t$ test to examine, at the $5 \%$ level of significance, whether the mean amount delivered by machine B may be taken as being greater than that delivered by machine A , stating carefully your null and alternative hypotheses and the required distributional assumption.
(b) Using the data for machine A in the table above, provide a two-sided $95 \%$ confidence interval for the mean amount delivered by this machine, stating the required distributional assumption. Explain whether you would conclude that the machine appears to be working correctly in terms of the nominal amount as set.

4 Quality control inspectors in a factory are investigating the lengths of glass tubes that will be used to make laboratory equipment.
(i) Data on the observed lengths of a random sample of 200 glass tubes from one batch are available in the form of a frequency distribution as follows.

| Length <br> $x(\mathrm{~mm})$ | Observed <br> frequency |
| :---: | :---: |
| $x \leqslant 298$ | 1 |
| $298<x \leqslant 300$ | 30 |
| $300<x \leqslant 301$ | 62 |
| $301<x \leqslant 302$ | 70 |
| $302<x \leqslant 304$ | 34 |
| $x>304$ | 3 |

The sample mean and standard deviation are 301.08 and 1.2655 respectively.
The corresponding expected frequencies for the Normal distribution with parameters estimated by the sample statistics are

| Length <br> $x(\mathrm{~mm})$ | Expected <br> frequency |
| :---: | :---: |
| $x \leqslant 298$ | 1.49 |
| $298<x \leqslant 300$ | 37.85 |
| $300<x \leqslant 301$ | 55.62 |
| $301<x \leqslant 302$ | 58.32 |
| $302<x \leqslant 304$ | 44.62 |
| $x>304$ | 2.10 |

Examine the goodness of fit of a Normal distribution, using a 5\% significance level.
(ii) It is thought that the lengths of tubes in another batch have an underlying distribution similar to that for the batch in part (i) but possibly with different location and dispersion parameters. A random sample of 10 tubes from this batch gives the following lengths (in mm).

$$
\begin{array}{llllllllll}
301.3 & 301.4 & 299.6 & 302.2 & 300.3 & 303.2 & 302.6 & 301.8 & 300.9 & 300.8
\end{array}
$$

(A) Discuss briefly whether it would be appropriate to use a $t$ test to examine a hypothesis about the population mean length for this batch.
(B) Use a Wilcoxon test to examine at the $10 \%$ significance level whether the population median length for this batch is 301 mm .

\begin{tabular}{|c|c|c|c|c|}
\hline Q1 \& \(\mathrm{F}(t)=1-\mathrm{e}^{-t / 3} \quad(\mathrm{t}>0)\) \& \& \& \\
\hline (i) \& \begin{tabular}{l}
For median \(m, \frac{1}{2}=1-\mathrm{e}^{-m / 3}\)
\[
\begin{aligned}
\& \therefore \mathrm{e}^{-m / 3}=\frac{1}{2} \Rightarrow-\frac{m}{3}=\ln \frac{1}{2}=-0.6931 \\
\& \Rightarrow m=2.079
\end{aligned}
\] \\
For \(90^{\text {th }}\) percentile \(p, 0.9=1-\mathrm{e}^{-p / 3}\)
\[
\begin{aligned}
\& \therefore \mathrm{e}^{-p / 3}=0.1 \Rightarrow-\frac{p}{3}=\ln 0.1=-2.3026 \\
\& \Rightarrow p=6.908
\end{aligned}
\]
\end{tabular} \& \begin{tabular}{l}
M1 \\
M1 \\
A1 \\
M1 \\
A1
\end{tabular} \& attempt to solve, here or for 90th percentile. Depends on previous M mark. \& 5 \\
\hline (ii) \&  \& \begin{tabular}{l}
M1 \\
A1 \\
M1 \\
M1 \\
A1
\end{tabular} \& \begin{tabular}{l}
(for \(t>0\), but condone absence of this) \\
Quoting standard result gets \(0 / 3\) for the mean. \\
attempt to integrate by parts
\end{tabular} \& 5 \\
\hline (iii \& \[
\begin{aligned}
\mathrm{P}(T>\mu)= \& {[\text { from cdf }] \mathrm{e}^{-\mu / 3}=\mathrm{e}^{-1} } \\
\& =0.3679
\end{aligned}
\] \& \begin{tabular}{l}
M1 \\
A1
\end{tabular} \& \begin{tabular}{l}
[or via pdf] \\
ft c's mean (>0)
\end{tabular} \& 2 \\
\hline (iv) \& \(\bar{T} \sim(\) approx \() \mathrm{N}\left(3, \frac{9}{30}=0.3\right)\) \& \[
\begin{aligned}
\& \text { B1 } \\
\& \text { B1 } \\
\& \text { B1 }
\end{aligned}
\] \& \[
\begin{aligned}
\& \mathrm{N} \\
\& \mathrm{ftc} \text { c's mean }(>0) \\
\& 0.3
\end{aligned}
\] \& 3 \\
\hline (v) \& \begin{tabular}{l}
EITHER can argue that 4.2 is more than 2 \\
SDs from \(\mu\)
\[
(3+2 \sqrt{0.3}=4.095 ;
\] \\
must refer to \(\mathrm{SD}(\overline{\mathrm{T}})\), not \(\operatorname{SD}(\mathrm{T})\) ) \\
i.e. outlier
\[
\Rightarrow \text { doubt }
\] \\
OR \\
significance test: \\
\(\frac{4.2-3}{3 / \sqrt{30}}=2.191\), refer to \(\mathrm{N}(0,1)\), sig at (eg) \(5 \%\)
\[
\Rightarrow \text { doubt }
\]
\end{tabular} \& \begin{tabular}{l}
M1 \\
M1 \\
A1 \\
M1 \\
M1 \\
A1
\end{tabular} \& Depends on first M, but could imply it. \& 3

18 <br>
\hline
\end{tabular}

| Q2 | $X \sim \mathrm{~N}(180, \sigma=12)$ |  | When a candidate's answers suggest that (s)he appears to have neglected to use the difference columns of the Normal distribution tables penalise the first occurrence only. |  |
| :---: | :---: | :---: | :---: | :---: |
| (i) | $\begin{aligned} & \mathrm{P}(X<170)=\mathrm{P}\left(Z<\frac{170-180}{12}=-0.8333\right) \\ & =1-0.7976=0.2024 \end{aligned}$ | M <br> A1 <br> A1 | For standardising. Award once, here or elsewhere. | 3 |
| (ii) | $\begin{aligned} & X_{1}+X_{2}+X_{3}+X_{4}+X_{5} \sim \mathrm{~N}\left(900, \sigma^{2}=720[\sigma=26.8328]\right. \\ & \mathrm{P}(\text { this }<840)=\mathrm{P}\left(Z<\frac{840-900}{26.8328}=-2.236\right) \\ & =1-0.9873=0.0127 \end{aligned}$ | B1 <br> B1 <br> A1 | Mean. <br> Variance. Accept sd. c.a.o. | 3 |
| $\begin{aligned} & \text { (iii } \\ & )^{\prime} \end{aligned}$ | $\begin{aligned} & Y \sim \mathrm{~N}(50, \sigma=6) \\ & X+Y \sim \mathrm{~N}\left(230, \sigma^{2}=180[\sigma=13.4164]\right) \\ & \mathrm{P}(\text { this }>240)=\mathrm{P}\left(Z>\frac{240-230}{13.4164}=0.7454\right) \\ & =1-0.7720=0.2280 \end{aligned}$ | B1 <br> B1 <br> A1 | Mean. <br> Variance. Accept sd. <br> c.a.o. | 3 |
| (iv) | $\frac{1}{4} X \sim N\left(45, \sigma^{2}=\frac{1}{16} \times 144=9[\sigma=3]\right)$ <br> Require $t$ such that $\begin{aligned} 0.9 & =\mathrm{P}(\text { this }<t)=\mathrm{P}\left(Z<\frac{t-45}{3}\right)=\mathrm{P}(Z<1.282) \\ \therefore t-45 & =3 \times 1.282 \Rightarrow t=48.85(48.846) \end{aligned}$ | B1 <br> M <br> B1 <br> A1 | Variance. Accept sd. FT incorrect mean. Formulation of requirement. $1.282$ <br> ft only for incorrect mean | 4 |
| (v) | $\begin{aligned} & I=45+T \text { where } T \sim \mathrm{~N}(120, \sigma=10) \\ & \therefore I \sim \mathrm{~N}(165, \sigma=10) \\ & \mathrm{P}(I<150)=\mathrm{P}\left(Z<\frac{150-165}{10}=-1.5\right) \\ & =1-0.9332=0.0668 \end{aligned}$ | B1 A1 | for unchanged $\sigma$ (candidates might work with $\mathrm{P}(T<105))$ <br> c.a.o. | 2 |
| (vi) | $J=30+\frac{3}{5} T \text { where } T \sim \mathrm{~N}(120, \sigma=10)$ |  | Cands might work with $\begin{aligned} & \mathrm{P}\left(\frac{3}{5} T<75\right) . \\ & \frac{3}{5} T \sim N(72,36) \end{aligned}$ |  |






| (B) | Data <br> 301.3 <br> 301.4 <br> 299.6 <br> 302.2 <br> 300.3 <br> 303.2 <br> 302.6 <br> 301.8 <br> 300.9 <br> 300.8$T=1+2$ <br> 39) <br> Refer to <br> (/paired) <br> Lower (or <br> needed <br> Value fo <br> Result is <br> No evid | Median 301 $\qquad$ $\square$ $\qquad$ $\square$ $\qquad$ $5+8=1$ <br> les of W atistic upper if 3 $n=10$ is t signific against | Difference <br>  <br> 0.3 <br> 0.4 <br> -1.4 <br> 1.2 <br> -0.7 <br> 2.2 <br> 1.6 <br> 0.8 <br> -0.1 <br> -0.2 <br> (or $3+4+6+$ <br> coxon singl <br> used) $5 \%$ t <br> (or 45 if 39 nt median being | Rank of <br> \|diff| <br> 3 <br> 4 <br> 8 <br> 7 <br> 5 <br> 10 <br> 9 <br> 6 <br> 1 <br> 2 <br> $7+9+10=$ sample 1 is used) 301 | M <br> M <br> A1 <br> B1 <br> M <br> M <br> A1 <br> E1 <br> E1 | for differences. ZERO in this section if differences not used. <br> for ranks. FT if ranks wrong. | 9 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  |  |  |  |  |  | 18 |

## 4768: Statistics 3

## General Comments

The overall standard of the scripts seen was pleasing: many candidates appeared well prepared for this paper. However, the quality of their comments, interpretations and explanations was consistently below that of the rest of the work.
Invariably all four questions were attempted. Question 2 was found to be very accessible and most candidates scored full or nearly full marks. Questions 1, 3 and 4 were well answered, with many candidates scoring relatively high marks. There was no evidence to suggest that candidates found themselves short of time at the end.

## Comments on Individual Questions

1) Continuous random variables; exponential distribution; Central Limit Theorem; delays at a railway junction.
(i) This part of the question was almost always answered correctly.
(ii) Most candidates found the pdf correctly, though occasional errors with the differentiation of the exponential function were seen.
By contrast quite a few candidates got into difficulty over finding the mean delay, often because they did not seem able to handle the integration by parts successfully. There were many candidates who sailed through this part with apparent ease.
(iii) By and large candidates seemed to be aware that in order to answer this part they needed to use the cdf and work out $1-\mathrm{F}(\mu)$.
(iv) This part of the question was not answered as well as might have been expected. There were some who thought that the mean of the distribution of $\bar{T}$ would be $\bar{T}$ itself, while others multiplied the mean and variance by 30 .
(v) It was expected that candidates would use the distribution of $\bar{T}$ from the previous part either to carry out a brief, informal hypothesis test or to consider the usual criterion (mean $\pm 2$ s.d.'s) for an outlier. In practice many candidates' answers fell somewhere between the two, which was quite acceptable.
2) 

## Combinations of Normal distributions; the times taken to set and check exam questions.

(i) This part was almost always answered correctly.
(ii) A few candidates worked out the variance here incorrectly as $5^{2} \times 12^{2}$. Other than that, this part was usually correct.
(iii) Only a few candidates experienced difficulty in getting this part right.
(iv) Candidates managed to get this part wrong in one of two ways. Either they worked out the variance as $12^{2} \div 4$, or they found and used the distribution of $1 / 4 Y$ (from part (iii)), believing it to be $1 / 4 X$. The value of $\Phi^{-1}(0.9)$ was almost always quoted and used correctly.
(v) Candidates seemed to have little difficulty with this part. Most preferred to work with $\mathrm{P}(T<105)$.
(vi) In this part most candidates preferred to work with $\mathrm{P}(3 T / 5<75)$, but some worked out the variance incorrectly as 60 rather than 36 .

The $\boldsymbol{t}$ distribution: paired hypothesis test for the difference between two population means; confidence interval for a population mean; production lines filling bottles of liquid soap.
(a) The hypotheses were usually stated satisfactorily but many candidates did not define their symbol $\mu$ adequately as the mean difference. Nor did they manage to convey that the required assumption was that the differences come from a Normal population.
The test statistic was usually worked out correctly from the data. Very few candidates failed to recognise that this was a paired comparison. The correct $t$ distribution was usually used for the test, and the only blemish was in the final conclusion: candidates did not take care to express this carefully enough, including mentioning that it is the mean that has been tested.
(b) By and large the confidence interval was found correctly, but there was a smattering of the types of errors that one might anticipate such as the wrong percentage point. Many candidates gave a satisfactory explanation about the correct working of the machine with regard to the nominal setting, although quite a few felt it necessary to quote a generic interpretation of a confidence interval. Candidates needed to take care over the required assumption here. In a question like this, including part (a) above, it is important to be clear about which population is required to be Normal.

Chi-squared hypothesis test for the goodness of fit of a Normal model; Wilcoxon test for a population median; manufacture of glass tubes.
(i) There was some uncertainty about the need for combining classes in this question. Nonetheless the correct test statistic was obtained in the majority of cases.
Many candidates stated the number of degrees of freedom incorrectly and hence looked up the wrong critical value. This was usually because they did not allow for the estimated parameters (the mean and variance) and/or for having combined classes.

It appeared that a lot of candidates did not read this part of the question carefully enough. They may have known that a Normal population is required for a $t$ test, but they certainly did not appreciate the significance of referring back to their conclusion to part (i) for guidance.
(B) The Wilcoxon test was carried out very competently by many candidates. Occasionally there was some carelessness in working out the differences from the stated median and this resulted in incorrect rankings. Sometimes the final part of the conclusion, which is expected to be in context, was expressed poorly.

## Grade Thresholds

## Jan 2006 Examination Series

Unit Threshold Marks

| Unit |  | Maximum <br> mark | A | B | C | D | E | U |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| All units | UMS | 100 | 80 | 70 | 60 | 50 | 40 | 0 |
| $\mathbf{4 7 6 8}$ | Raw | 72 | 56 | 48 | 41 | 34 | 27 | 0 |

