Oxford Cambridge and RSA

Tuesday 21 June 2016 – Morning

AS GCE MATHEMATICS (MEI)

4776/01 Numerical Methods

QUESTION PAPER

Candidates answer on the Printed Answer Book.

OCR supplied materials:
- Printed Answer Book 4776/01
- MEI Examination Formulae and Tables (MF2)

Other materials required:
- Scientific or graphical calculator

Duration: 1 hour 30 minutes

INSTRUCTIONS TO CANDIDATES

These instructions are the same on the Printed Answer Book and the Question Paper.

- The Question Paper will be found inside the Printed Answer Book.
- Write your name, centre number and candidate number in the spaces provided on the Printed Answer Book. Please write clearly and in capital letters.
- **Write your answer to each question in the space provided in the Printed Answer Book**. Additional paper may be used if necessary but you must clearly show your candidate number, centre number and question number(s).
- Use black ink. HB pencil may be used for graphs and diagrams only.
- Read each question carefully. Make sure you know what you have to do before starting your answer.
- Answer **all** the questions.
- Do **not** write in the bar codes.
- You are permitted to use a scientific or graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.

INFORMATION FOR CANDIDATES

This information is the same on the Printed Answer Book and the Question Paper.

- The number of marks is given in brackets **[ ]** at the end of each question or part question on the Question Paper.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.
- The total number of marks for this paper is **72**.
- The Printed Answer Book consists of **12** pages. The Question Paper consists of **8** pages. Any blank pages are indicated.

INSTRUCTION TO EXAMS OFFICER / INVIGILATOR

- Do not send this Question Paper for marking; it should be retained in the centre or recycled. Please contact OCR Copyright should you wish to re-use this document.

Section A (36 marks)

1 The expression $\sqrt{\frac{n}{n-1}}$ is sometimes approximated by 1 when n is large.

(i) Find the absolute and the relative error in this approximation when $n = 40$. **[2]**

(ii) Using trial and error or otherwise, find the smallest integer n for which the magnitude of the relative error is less than 1%. **[2]**

2 You are given that the equation

$$x^3 + x - 3 = 0$$

has a single real root α, where $1 < \alpha < 2$.

(i) Use the Newton-Raphson method with $x_0 = 1.5$ to find α correct to 5 decimal places. **[5]**

(ii) By considering ratios of differences, show that the Newton-Raphson method is faster than first order. **[3]**

3 A function f(x) has the following values, correct to 5 decimal places. (The values of x are exact.)

x	0.5	1	1.5	2	2.5
f(x)	0.958 85	0.841 47	0.665 00	0.454 65	0.239 39

(i) Obtain two Simpson's rule estimates of $I = \int_{0.5}^{2.5} \mathrm{f}(x)\,\mathrm{d}x$. **[3]**

(ii) State the order of Simpson's rule and hence estimate the value of I that would be obtained if f(x) were known at $x = 0.75, 1.25, 1.75, 2.25$. **[4]**

(iii) Give the value of I to the accuracy that is justified. **[1]**

4 **(i)** State the orders of accuracy of the forward difference and central difference formulae for numerical differentiation. Explain what this means in practice. **[3]**

(ii) A function $g(x)$ has the following values, correct to 5 decimal places. (The values of x are exact.)

x	-0.2	-0.1	0	0.1	0.2
$g(x)$	0.755 60	0.876 86	1	1.123 14	1.244 40

Obtain two estimates of $g'(0)$ using the forward difference formula, and two estimates of $g'(0)$ using the central difference formula.

Comment on your estimates. **[5]**

5 A function $h(x)$ has values as shown in the table.

x	$h(x)$
0	1.357 01
0.5	1.413 33
1	1.381 77
1.5	1.264 31

(i) Show, by means of a difference table, that $h(x)$ can be well approximated by a quadratic. **[3]**

(ii) Use Newton's forward difference interpolation formula with $x_0 = 0$ to write down an expression for the quadratic approximation to $h(x)$. (You do not need to simplify this expression.) **[3]**

(iii) Find the error in the quadratic approximation at $x = 1.5$. **[2]**

Section B (36 marks)

6 **(i)** Show, by means of a sketch graph, that the equation

$$kx = 3^{-x}, \quad (*)$$

where $k > 0$, has exactly one root. **[4]**

(ii) Show numerically that the iterative formula

$$x_{r+1} = \tfrac{1}{k}3^{-x_r}, \quad (**)$$

with $x_0 = 1$,

(A) converges in the case $k = 0.5$,

(B) diverges in the case $k = 0.4$.

Explain why it would *not* be a good idea to use (**) in the case $k = 0.5$. **[7]**

(iii) Show that (*) may be rearranged as

$$x = 0.5\left(x + \tfrac{1}{k}3^{-x}\right).$$

Use an iteration based on this rearrangement to find the root of (*), correct to 4 decimal places, in the cases

(A) $k = 0.5$,

(B) $k = 0.4$. **[7]**

7 Let S_n be the sum of the first n terms in the series

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \cdots. \quad (*)$$

It is known that S_n converges to a limit S as n tends to infinity. A spreadsheet is used to investigate the rate of convergence of S_n to S.

(i) The spreadsheet gives $S_{1000} = 0.692\,647$, hence $0.692\,65$ to 5 decimal places.

Find S_{1001} and S_{1002} correct to 5 decimal places. Comment on the rate of convergence of S_n. **[4]**

(ii) Show, by combining adjacent terms, that (*) may be written as

$$\tfrac{1}{2} + \tfrac{1}{12} + \cdots. \quad (**)$$

State the next two terms in this series. **[3]**

Let T_n be the sum of the first n terms of (**). A spreadsheet is used to investigate the rate of convergence of T_n.

(iii) Explain why T_{500} will be $0.692\,65$ correct to 5 decimal places.

Find T_{501} and T_{502} correct to 5 decimal places. Comment on the rate of convergence of T_n. **[4]**

An improved method for summing (**) is to add a 'correction term' as follows.

$$T_n + \tfrac{1}{4n+2} \quad (***)$$

(iv) Evaluate (***) correct to 5 decimal places for $n = 500$ and $n = 501$.

Comment on your answers. **[4]**

(v) Discuss briefly what your answers to parts **(i)**, **(iii)** and **(iv)** indicate about convergence when successive answers agree to a certain number of decimal places.

Explain which, if any, of the sums calculated you would regard as the value of S correct to 5 decimal places. **[3]**

END OF QUESTION PAPER

Section B (36 marks)

6 (i) **There is a spare copy of this graph paper below.**

Spare copy of graph paper for question 6 (i)

Question			Answer	Marks	Guidance	
1	(i)		<table><tr><td>n</td><td>exact</td><td>approx</td><td>error</td><td>rel error</td></tr><tr><td>40</td><td>1.012739</td><td>1</td><td>−0.012739</td><td>−0.012580</td></tr></table>	B1 B1 **[2]**	Error (with either sign) Relative error: must be negative unless absolutely clear that the opposite convention has been used. At least 3 dp required	
	(ii)		Trial and error gives n = 51	M1 A1 **[2]**	Evidence of a serious and sensible attempt	
2	(i)		f '(x) = $3x^2 + 1$ NR formula Iterations, e.g.: 1.5 1.258065 1.214705 1.213413 1.213412 Hence 1.21341 to 5 dp	B1 B1 M1A1 A1 **[5]**	 Soi First iteration Answer to 5 dp	
	(ii)		Differences and ratios of differences for this iteration are: −0.24194 −0.04336 −0.00129 −1.1E−06 0.179218 0.02981 0.000869 Comment that ratios decreasing (so not first order)	 M1A1 E1 **[3]**	 Allow 1 mark if differences only shown FT decreasing but incorrect ratios	
3	(i)		Using x = 0.5, 1.5 and 2.5: S = 1.286080 Using all values of x: S = 1.285453	M1A1 A1 **[3]**	M1 for correct formula For correct second S	
	(ii)		S is of order h^4, or fourth order Difference in S values is (−)0.000627 Reduce difference by a factor of 16 (Next difference is (−)0.000039) So next S is estimated as 1.285414	B1 B1 M1 A1 **[4]**	 May be implied FT an incorrect difference here	
	(iii)		1.2854 is secure; accept also 1.28541	B1 **[1]**		
4	(i)		Forward difference is first order (order h) Central difference is second order (order h^2) This (usually) means that central difference is more accurate than forward difference, and that central difference converges faster than forward difference.	B1 B1 B1 **[3]**	 Accept either statement for the mark	

Question			Answer	Marks	Guidance	
	(ii)		h forward central 0.2 1.2220 1.2220 0.1 1.2314 1.2314 In this case the central difference is the same as the forward difference (There is a point of inflection at $x = 1$ (not required))	M1A1 M1A1 E1 [5]	For forward differences For central differences	
5	**(i)**		0 1.35701 0.5 1.41333 0.05632 1 1.38177 −0.03156 −0.08788 1.5 1.26431 −0.11746 −0.08590 2nd differences approximately equal so quadratic a good fit	M1A1 E1 **[3]**		
	(ii)		$\mathrm{h}(x) = 1.35701$ $+ 0.05632\, x / 0.5$ $- 0.08788\, x\,(x - 0.5) / 2\,(0.5)^2$	M1 A1 A1 **[3]**	Need to see attempt at second term	
	(iii)		Substitute to get $\mathrm{h}(1.5) = 1.26233$ Hence error is $1.26233 - 1.26431 = -0.00198$	B1 B1 **[2]**	Follow through trivial errors Condone opposite sign convention	
6	**(i)**		Sketch of $y = kx$ Sketch of $y = 3^{-x}$ Some consideration or mention of varying k	G1 G2 E1 **[4]**	G1 shape; then G1 location if shape correct	
	(ii)	**(A)**	$k = 0.5$ gives iterates 1 0.666667 0.9615 0.695469 0.931551 0.718732 The interval is narrowing so the iteration is converging	M1A1 E1	Must be enough terms (at least 4) to see the pattern	
		(B)	$k = 0.4$ gives iterates 1 0.833333 1.000781 0.832619 1.001567 0.8319 The interval is getting wider so the iteration is diverging	M1A1 E1	Must be enough terms (at least 4) to see the pattern	
			The convergence with $k = 0.5$ is very slow	E1 **[7]**	Dependent on (A) being correct	

Question			Answer	Marks	Guidance	
	(iii)		Convincing algebra (in either direction) to show the equivalence k = 0.5 gives 1 0.833333 0.816979 0.816059 0.816011 0.816009 Hence root is 0.8160 to 4dp k = 0.4 gives 1 0.916667 0.914947 0.914951 0.914951 0.914951 Hence root is 0.9150 to 4dp	M1A1 M1A1 A1 B1 B1 **[7]**		
7	(i)		S_{1001} = 0.692647 + 1 / 1001 = 0.693646 = 0.69365 to 5 dp S_{1002} = 0.693646 – 1 / 1002 = 0.692648 = 0.69265 to 5 dp (Still variation in 3rd dp so) convergence is slow	M1A1 A1 E1 **[4]**	M1 for method in either evaluation 'slow' is the key word: the qn states the convergence	
	(ii)		Convincing arithmetic to 1/2 and 1/12 1/5 – 1/6 = 1/30 1/7 – 1/8 = 1/56	B1 B1 B1 **[3]**		
	(iii)		$T_{500} = S_{1000}$ $T_{501} = S_{1002}$ = 0.69265 to 5 dp $T_{502} = S_{1004}$ = 0.692648 + 1/1003 – 1/1004 = 0.692649 = 0.69265 to 5 dp T_n *appears* to be converging judging by agreement to 5 dp	E1 B1 B1 E1 **[4]**	Do not accept T_n is converging	
	(iv)		With correction, n = 500 gives 0.693147 so 0.69315 to 5 dp With correction, n = 501 gives 0.693147 so 0.69315 to 5 dp The corrected formula gives different values to those obtained earlier	M1A1 A1 E1 **[4]**	6 dp values not required	
	(v)		Agreement to (5) dp does not indicate convergence to (5) dp So be cautious about accepting any of the values calculated Though the last is likely to be more reliable than the others (because it's an improved method, or because it is in the middle of the oscillations)	E1 E1 E1 **[3]**	Accept other sensible discussion points	

4776 Numerical Methods (Written Examination)

General Comments:

Candidates seemed, for the most part, well prepared for this paper. Routine numerical work was generally done well, and the selection of methods was usually appropriate. Interpretation of results was less good, however, and where questions called for comments the answers given were quite often weak.

Poor presentation continues to be an issue, with illegible hand-writing and a propensity to scatter numerical work haphazardly on the page being far too common. There were several instances in which the work submitted simply couldn't be deciphered.

Comments on Individual Questions:

Question No. 1
Most candidates handled this question on absolute and relative errors well. In part (i), some candidates took the denominator in the relative error term to be the approximate value. In part (ii), some ignored the suggestion to use trial and error and attempted an algebraic approach; this proved far harder.

Question No. 2
Part (i) was a routine exercise in using the Newton-Raphson method, and it was generally done well. In part (ii) the approach asked for required the calculation of ratios of differences and the observation that they decrease rapidly. A fairly common error was to say that the ratio of differences in the Newton-Raphson method will be less than 0.5.

Question No. 3
Again, the straightforward numerical work in part (i) of this question was done well. In part (ii), most stated correctly that Simpson's rule is of order 4. However, using that fact to extrapolate to the next Simpson's rule estimate was found more difficult. In the final part, many candidates were overly cautious in stating the number of decimal places to which the integral is accurate.

Question No. 4
The orders of accuracy of the forward difference and central difference methods were usually stated correctly. In practice this means that the central difference method will usually converge more quickly than the forward difference method. In part (ii), the two methods give the same values as one another. Some candidates correctly observed that the function appears to have a point of inflection at $x = 0$, but this observation was not required for full marks.

Question No. 5
The difference table was constructed correctly by the vast majority, and they commented that the similarity of the second differences indicated that a quadratic would be a good fit. The forward difference interpolation formula was generally done well. It was encouraging to see far fewer examples than previously of candidates confusing the x and $f(x)$ values.

Question No. 6
This question proved to be a fairly straightforward exercise in fixed point iteration. In part (i) most candidates drew the two functions correctly. However it was rare to see any comment on how the graph of $y = kx$ depended on the value of k. In part (ii), some comment or discussion or explanation was required of the convergent and divergent processes. It was not judged sufficient simply to exhibit some iterates. Neither was it was sufficient to add the words 'converging' and 'diverging': the information is already in the question. The simplest approach was to note that the oscillating

iterates are getting closer in one case and further apart in the other. The algebra in part (iii) was done well, as were the two numerical calculations.

Question No. 7
This question was all about the relationship between convergence to an answer and agreement to a number of decimal places. Candidates carried out the numerical work correctly, but there were very few clear and correct comments on the underlying concepts. In part (i) it was evident from successive terms that the sequence S_n is converging slowly. In parts (ii) and (iii), the same sequence is explored, but now with adjacent terms combined. This gives the *impression* of convergence because of the agreement of decimal places. However, we know that the convergence can't be any faster because the sequence is essentially the same as in part (i). In part (iv) a correction term is added to improve the convergence. This gives agreement to 5 decimal places, but we should be hesitant in saying that the answer is correct to 5 decimal places as we have already seen that agreement does not imply convergence. Part (v) asks candidates to comment on convergence and agreement in the light of their earlier answers. Unfortunately, most candidates made incorrect statements here: despite the evidence to the contrary, they often said that agreement indicates that a sequence has converged.