## ADVANCED SUBSIDIARY GCE <br> MATHEMATICS (MEI)

4776/01
Numerical Methods

## Candidates answer on the Answer Booklet

## OCR Supplied Materials:

- 8 page Answer Booklet
- Graph paper
- MEI Examination Formulae and Tables (MF2)


## Other Materials Required:

None
Friday 15 January 2010
Afternoon
Duration: 1 hour 30 minutes


## INSTRUCTIONS TO CANDIDATES

- Write your name clearly in capital letters, your Centre Number and Candidate Number in the spaces provided on the Answer Booklet.
- Use black ink. Pencil may be used for graphs and diagrams only.
- Read each question carefully and make sure that you know what you have to do before starting your answer.
- Answer all the questions.
- Do not write in the bar codes.
- You are permitted to use a graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.


## INFORMATION FOR CANDIDATES

- The number of marks is given in brackets [ ] at the end of each question or part question.
- You are advised that an answer may receive no marks unless you show sufficient detail of the working to indicate that a correct method is being used.
- The total number of marks for this paper is 72.
- This document consists of $\mathbf{4}$ pages. Any blank pages are indicated.


## Section A (36 marks)

1 Show that the equation

$$
2^{x}+\left(\frac{1}{2}\right)^{x}=3
$$

has a root between $x=1.3$ and $x=1.5$. Use the bisection method to find an estimate of this root with a maximum possible error less than 0.02 .

Determine how many further iterations would be required to reduce the maximum possible error to less than 0.001 .

2 An integral, $\int_{a}^{b} \mathrm{f}(x) \mathrm{d} x$, is being evaluated numerically. Some mid-point rule and trapezium rule estimates are shown in the table.

| $h$ | Mid-point rule | Trapezium rule |
| :---: | :--- | :--- |
| 1 | 2.579768 | 2.447490 |
| 0.5 | 2.547350 |  |

Find the trapezium rule estimate for $h=0.5$.
Find two Simpson's rule estimates and hence state, with a reason, the value of the integral to the accuracy that appears justified.

3 (i) Given that $\mathrm{f}(x)=x^{3}-x^{2}+1$, find $\mathrm{f}(0.5)$.
Use the formula $\mathrm{f}(x+h) \approx \mathrm{f}(x)+h \mathrm{f}^{\prime}(x)$ to show that

$$
\begin{equation*}
\mathrm{f}(0.5+h) \approx 0.875-0.25 h . \tag{3}
\end{equation*}
$$

(ii) Hence determine the approximate range of values of $x$ for which $\mathrm{f}(x)=0.875$ correct to 3 decimal places.

4 (i) Show algebraically that

$$
\begin{equation*}
(k+1)^{2}+(k-1)^{2}-2 k^{2}=2 \tag{*}
\end{equation*}
$$

for all values of $k$.
(ii) Use your calculator to evaluate the left hand side of (*) for increasingly large values of $k$ (e.g. $10^{3}$, $\left.10^{6}, 10^{9}, \ldots\right)$. State briefly two important results in numerical methods that are illustrated by your working.

5 A function $\mathrm{f}(x)$ has the following values correct to 3 decimal places.

| $x$ | 0 | 1 | 2 | 3 | 4 |
| :---: | :--- | :--- | :--- | :--- | :--- |
| $\mathrm{f}(x)$ | 1.883 | 2.342 | 2.874 | 3.491 | 4.206 |

(i) Show, by means of a difference table, that a cubic polynomial fits these data points closely but not exactly.
(ii) Use Newton's forward difference formula to estimate the value of $\mathrm{f}(1.5)$.

## Section B (36 marks)

6 (i) The derivative of a function is to be estimated numerically. Show, with the aid of a sketch, that the central difference method will generally be more accurate than the forward difference method.
(ii) The table shows two values of $\tan x^{\circ}$ correct to 7 significant figures.

| $x$ | 60 | 62 |
| :---: | :--- | :--- |
| $\tan x^{\circ}$ | 1.732051 | 1.880726 |

Use these two values to estimate the derivative of $\tan x^{\circ}$ at $x=60$.
Use your calculator to find two further estimates of this derivative, using the forward difference method and taking $h=1$ and $h=0.5$.
(iii) Use the central difference method with $h=2, h=1$ and $h=0.5$ to obtain three estimates of the derivative of $\tan x^{0}$ at $x=60$.
(iv) Show that the differences between the estimates in part (ii) reduce by a factor of about 0.5 as $h$ is halved.

By considering the differences between the estimates in part (iii) show that the central difference method seems to converge more rapidly than the forward difference method.
[Question 7 is printed overleaf.]

7 (i) Show, by means of a sketch or otherwise, that the equation

$$
\begin{equation*}
x=3 \sin x, \tag{*}
\end{equation*}
$$

where $x$ is in radians, has a root, $\alpha$, in the interval $\left(\frac{1}{2} \pi, \pi\right)$. Determine how many other non-zero roots, if any, the equation has.
(ii) Determine whether or not the iteration

$$
x_{r+1}=3 \sin x_{r},
$$

starting with $x_{0}=2$, converges to $\alpha$. Illustrate your answer with a staircase or cobweb diagram as appropriate.
(iii) Show that equation (*) may be rearranged into the form

$$
x=\sin x+\frac{2}{3} x
$$

Show that the corresponding iteration, starting with $x_{0}=2$, converges rapidly. State to 5 decimal places the value to which the iteration converges. Verify that this value for $\alpha$ is correct to 5 decimal places.

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## 4776 Numerical Methods

1

| $x$ | LHS |  |
| ---: | ---: | ---: |
| 1.3 | 2.868415 | $<3$ |
| 1.5 | 3.181981 | $>3$ |


| 1.4 | 3.017945 | mpe | (may be implied) |  |
| ---: | ---: | ---: | :--- | :--- |
| 1.35 | 2.941413 | 0.1 |  |  |
| 1.375 | 2.979232 | 0.05 |  | finishing at this point: |
| 1.3875 |  | 0.025 |  |  |
|  |  | 0.0125 |  |  |

$1.35 \quad 2.941413$
mpe: $\quad 0.006250 .0031250 .0015630 .000781<0.001$ so 4 more iterations [M1A1]

| $\mathbf{2}$ | $h$ | $M$ | $T$ | $S$ |
| :---: | :---: | :---: | :--- | :--- |
|  | 1 | 2.579768 | 2.447490 | $\mathbf{2 . 5 3 5 6 7 5}$ |

$T \quad$ [M1A1]
$S$ [M1A1A1]
2.536 secure by comparison of $S$ values.

3(i) $\quad \mathrm{f}^{\prime}(x)=3 x^{2}-2 x \quad$ so $\mathrm{f}^{\prime}(0.5)=-0.25$
$\mathrm{f}(0.5)=0.875$ hence given result
[B1]
(ii) Require $-0.0005<0.25 h<0.0005$

Hence $-0.002<h<0.002$
And so $0.498<x<0.502$

4(i) Convincing algebra to given result
[M1A1]
(ii) Eg $k=1000$ correct evaluation to 2 ..... [B1]
$k=1000000$ incorrect evaluation to zero (NB some will need larger $k$ ) ..... [B1]

Mathematically equivalent expressions do not always evaluate equally

(because calculators do not store (large) numbers exactly)

Subtraction of nearly equal quantities often causes problems

| 5(i) | $x$ | $\mathrm{f}(x)$ | $\Delta \mathrm{f}(\mathrm{x})$ | $\Delta^{2} \mathrm{f}(x)$ | $\Delta^{3} \mathrm{f}(x)$ | 1st diff: <br> 2nd, 3rd | [M1A1] |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | 0 | 1.883 |  |  |  |  |  |
|  | 1 | 2.342 | 0.459 |  |  |  | [F1] |
|  | 2 | 2.874 | 0.532 | 0.073 |  |  |  |
|  | 3 | 3.491 | 0.617 | 0.085 | 0.012 |  | [E1] |
|  | 4 | 4.206 | 0.715 | 0.098 | 0.013 |  |  |
| 4.206 3rd diffs almost constant |  |  |  |  |  |  |  |

(ii) $\mathrm{f}(1.5)=1.883+0.459 \times 1.5+0.073 \times 1.5 \times 0.5 / 2!+0.012 \times 1.5 \times 0.5 \times(-0.5) / 3$ !

6 (i) Sketch of smooth curve and its tangent.
Forward and central difference chords.
Clear statement or implication that the central difference chord has gradient closer to that of the tangent
(ii)

| $h$ | $\tan 60^{\circ}$ |
| ---: | ---: |
| 2 | 1.732051 |
| 1 | 1.732051 |
| 0.5 | 1.732051 |

$\tan (60+h)^{\circ}$
1.880726
1.804048
1.767494
derivative
0.074338
0.071997
0.070886
[M1A1]
[A1]
[A1]
[subtotal 4]
(iii) $h$

| $\tan (60+h)^{\circ}$ | $\tan (60-h)^{\circ}$ |
| ---: | :---: |
| 1.880726 | 1.600335 |
| 1.804048 | 1.664279 |
| 1.767494 | 1.697663 |

derivative
0.070098
0.069884
0.069831
[M1A1]
[A1]
[A1]
(iv) forward difference: derivative diffs ratio of diffs
$\begin{array}{llll}0.074338 & & & \\ 0.071997 & -0.00234 & & \\ 0.070886 & -0.00111 & 0.474407 & \text { (about 0.5, may be implied) }\end{array}$
[M1A1A1]
central difference: derivative diffs ratio of diffs
0.070098
$0.069884-0.00021$
$0.069831-5.3 \mathrm{E}-05 \quad 0.24896 \quad$ (about 0.25 , less than

7 (i) Sketch showing $y=3 \sin x$ and $y=x$ with intersection in $(1 / 2 \pi, \pi)$
(ii)

| $r$ | 0 | 1 | 2 | 3 | 4 |
| ---: | :--- | ---: | ---: | ---: | ---: |
| $x_{r}$ | 2 | 2.727892 | 1.206001 | 2.80259 | 0.997639 |
|  | clearly not converging | 2.52058 |  |  |  |
|  | Cobweb diagram to illustrate process |  |  |  |  |
|  |  |  |  |  |  |

(iii) Convincing algebra to given result.

| $r$ | 0 | 1 | 2 | 3 | 4 | 5 |
| ---: | ---: | ---: | ---: | ---: | ---: | ---: |
| $x_{r}$ | 2 | 2.242631 | 2.277768 | 2.278844 | 2.278862 | 2.278863 |

[M1A1A1]
Root appears to be 2.27886 to 5 dp

| $x$ | $\sin x+2 / 3 x$ |
| ---: | :--- |
| 2.278855 |  |
| 2.278865 | $>2.2788625$ |
| 2.2788627 |  |$\quad$ hence result is correct to 5 dp

