OXFORD CAMBRIDGE AND RSA EXAMINATIONS

Advanced Subsidiary General Certificate of Education
Advanced General Certificate of Education

MEI STRUCTURED MATHEMATICS 4776

Numerical Methods

Wednesday **25 JANUARY 2006** Morning 1 hour 30 minutes

Additional materials:
8 page answer booklet
Graph paper
MEI Examination Formulae and Tables (MF2)

TIME 1 hour 30 minutes

INSTRUCTIONS TO CANDIDATES

- Write your name, centre number and candidate number in the spaces provided on the answer booklet.
- Answer **all** the questions.
- You are permitted to use a graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets [ ] at the end of each question or part question.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.
- The total number of marks for this paper is 72.

This question paper consists of 4 printed pages.

Section A (36 marks)

1 Show that if

$$X = x(1 + r)$$

then

$$\frac{1}{X} \approx \frac{1}{x}(1 - r)$$

for sufficiently small r.

Explain carefully what this means in terms of relative errors. Illustrate your answer in the case $X = 10$, $x = 9.8$. [6]

2 The equation

$$\frac{1}{x} = \sin x \qquad (*)$$

(where x is in radians) has two roots, α and β, in the interval $0 < x < \pi$.

(i) Given that $\alpha \approx 1$, use an iteration based on a simple rearrangement of (*) to find α correct to 3 significant figures. [4]

(ii) Verify that $\beta = 2.773$ is correct to 3 decimal places. [3]

3 The integral $I = \int_1^3 \mathrm{f}(x)\,\mathrm{d}x$ is being evaluated numerically. The following values have been obtained.

h	mid-point rule	trapezium rule
2	2.602 42	2.448 66
1	2.569 82	

Obtain 2 further trapezium rule estimates of I.

Obtain 2 Simpson's rule estimates of I.

Give a value for I to the accuracy that appears justified. [7]

4 The function $f(x)$ has the values shown in the table.

x	1	2	3	4	5
$f(x)$	3.0	4.5	5.4	6.2	6.7

(i) Obtain two estimates of the gradient $f'(3)$ using the central difference method. [3]

(ii) Given that the values of x are exact but that the values of $f(x)$ are rounded to 1 decimal place, determine a range of possible values of $f'(3)$ from each estimate. Comment on your results.

Give a value for $f'(3)$ to the accuracy that appears justified. [5]

5 The function $g(x)$ is quadratic. The following values are known.

x	1	3	4
$g(x)$	4	1	11

Use Lagrange's method to determine $g(2)$.

Check your answer by drawing up a difference table for $g(x)$. [8]

Section B (36 marks)

6 **(i)** Show that the curve $y = x^{10} - 10x + 1$ has exactly one turning point.

Show that the equation

$$x^{10} - 10x + 1 = 0 \qquad (*)$$

has exactly two real roots and that these roots both lie in the interval [0, 2]. [7]

(ii) Use the Newton-Raphson method to find the larger root correct to 4 decimal places. [6]

(iii) Obtain another iteration based on a rearrangement of (*) and hence, *without using a calculator*, show that the smaller root is almost exactly $0.1 + 0.1^{11}$. [5]

[Turn over

7 The number e satisfies the relationship $\log_e N = \int_1^N \frac{1}{x}\,dx$. Hence it follows, for example, that $\log_e 5 = \int_1^5 \frac{1}{x}\,dx$.

(i) Use the mid-point rule to show that $\log_e 5 \approx \frac{1}{1.5} + \frac{1}{2.5} + \frac{1}{3.5} + \frac{1}{4.5}$. (*)

Given that, correct to 8 decimal places, $\log_e 5 = 1.609\,437\,91$, find the error in the approximation (*). [5]

(ii) Correct to 8 decimal places, $\log_e 2 = 0.693\ 147\ 18$. Hence or otherwise obtain the values of $\log_e 10$, $\log_e 20$, $\log_e 40$ and $\log_e 80$, giving your answers correct to 6 decimal places. [3]

It is known that, as N increases, the value of the expression

$$\log_e N - \left(\frac{1}{1.5} + \frac{1}{2.5} + \frac{1}{3.5} + \ldots + \frac{1}{(N-0.5)}\right)$$

tends to a constant k.

(iii) Use the information in the table to obtain 4 estimates, α_1, α_2, α_3, α_4, of k. Give these estimates to 6 decimal places.

N	10	20	40	80
$\frac{1}{1.5} + \frac{1}{2.5} + \frac{1}{3.5} + \ldots + \frac{1}{(N-0.5)}$	2.266511	2.959346	3.652416	4.345543

[3]

(iv) Find the differences $\alpha_2 - \alpha_1$, $\alpha_3 - \alpha_2$, $\alpha_4 - \alpha_3$. Show that the differences reduce by a factor of approximately 4 as N is doubled. Hence obtain the best estimate you can of k, giving your answer to an appropriate number of significant figures. [7]

MEI Numerical Methods 4776

1 Use binomial expansion of $(1 + r)^{-1}$ or sum of GP, or $1 - r^2$ with r^2 taken to be zero to obtain given result. **[M1A1]**
Relative error in reciprocal is of same magnitude but opposite sign **[E1E1]**
E.g. 10 is approx 2% greater than 9.8
1/10 = 0.1 is approx 2% less than 1/9.8 = 0.10204 **[M1A1]**
[TOTAL 6]

2(i) $x_{r+1} = 1/\sin(x_r)$ **[M1]**

r	0	1	2	3	10	11	12
x_r	1	1.188395	1.077852	1.135147	1.113855	1.114323	1.114067

[M1A1]

root is 1.11 to 3 sf. **[A1]**
[subtotal 4]

(ii)

x	2.7725	2.7735
1/x - sin(x)	-8.4E-05	0.000719

change of sign, so 2.773 correct to 3 dp **[M1A1A1]**
[subtotal 3]
[TOTAL 7]

3

h	M	T
2	2.60242	2.44866
1	2.56982	

T2 = (M1 + T1)/2 = 2.52554 **[M1A1]**
T4 = (M2 + T2)/2 = 2.54768 **[A1]**
S1 = (2M1 + T1)/3 = 2.55117 **[M1A1]**
S2 = (2M2 + T2)/3 = 2.55506 **[A1]**

I = 2.56 (or 2.555) is justified **[A1]**
[TOTAL 7]

4

x	1	2	3	4	5
f(*x*)	3	4.5	5.4	6.2	6.7

	(i)	**(ii)**	
h	f '(3)	min	max
2	0.925	0.9	0.95
1	0.85	0.8	0.9

(i) **[M1A1A1]**
(ii) **[M1A1A1]**

Larger h gives smaller interval (or 0.9 is the only common value) **[E1]**
f '(3) = 0.9 is the value that seems justified. **[E1]**
(Or 0.8 seems to be the limit the process is tending to. [E1E1])
[TOTAL 8]

5

x	1	3	4
$g(x)$	4	1	11

$L(2) = 4(2-3)(2-4)/(1-3)(1-4) + 1(2-1)(2-4)/(3-1)(3-4) + 11(2-1)(2-3)/(4-1)(4-3)$ **[M1A1A1A1]**

$= -1\,^1/_3$ **[A1]**

x	g(x)	Δg(x)	Δ^2g(x)
1	4	-5.33333	7.666666
2	-1.33333	2.333333	7.666667
3	1	10	
4	11		

2nd differences constant so correct for a quadratic **[M1A1E1]**

[TOTAL 8]

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6 (i) $y' = 10x^9 - 10 = 0$ only when $x = 1$, hence at most one turning point **[M1A1]**

tenth degree polynomial is positive as x tends to plus or minus infinity **[E1]**

(hence exactly one turning point) (or other methods)

x	0	1	2
f(x)	1	-8	1005

[M1A1]

changes of sign so roots in [0,1] and [1,2] **[E1]**

since only one turning point cannot be any more roots **[E1]**

[subtotal 7]

(ii) NR: $x_{r+1} = x_r - (x_r^{10} - 10x_r + 1)/(10x_r^9 - 10)$ **[M1A1A1]**

r	0	1	2	3	4	5
x_r	1.2	1.315589	1.284353	1.280004	1.279928	1.279928
						1.2799

[M1A1A1]

[subtotal 6]

(iii) $x_{r+1} = (x_r^{10} + 1)/10$ **[M1A1]**

If $x_0 = 0.1$ then $x_1 = (0.1^{10} + 1)/10 = 0.1^{11} + 0.1$ **[M1A1]**

This is so close to 0.1 that further iterations are unnecessary **[E1]**

[subtotal 5]

[TOTAL 18]

7 (i) Mid-point rule with h=1 and 4 strips to obtain given result. **[M1A1]**

$\log_e(5)$ =	1.609438			
Mid-pt =	1.574603	error is (-)	0.034835	**[M1A1A1]**
				[subtotal 5]

(ii)

N	10	20	40	80	
$\log_e(N)$	2.302585	2.995732	3.688879	4.382027	**[M1A1A1]**
					[subtotal 3]

(iii)(iv)

N	Mid-pt	ln(N)	est k	diffs	ratio of diffs		estimates
10	2.266511	2.302585	0.036074				**[M1A1A1]**
20	2.959346	2.995732	0.036386	0.000312			***[subtotal 3]***
40	3.652416	3.688879	0.036463	7.72E-05	0.247231	**diffs**	**[M1A1]**
80	4.345543	4.382027	0.036484	2.02E-05	0.261472	**ratio**	**[M1A1]**
					(approx 0.25)		

extrapolating:	0.036489	5.05E-06	
(or equivalent)	0.036490	1.26E-06	
	0.036490	3.15E-07	**[M1A1]**

0.03649(0) seems secure **[A1]**

[subtotal 7]

[TOTAL 18]