OCR
RECOGNISING ACHIEVEMENT

OXFORD CAMBRIDGE AND RSA EXAMINATIONS

Advanced Subsidiary General Certificate of Education
Advanced General Certificate of Education

MEI STRUCTURED MATHEMATICS 4776

Numerical Methods

Wednesday **25 MAY 2005** Afternoon 1 hour 30 minutes

Additional materials:
Answer booklet
Graph paper
MEI Examination Formulae and Tables (MF2)

TIME 1 hour 30 minutes

INSTRUCTIONS TO CANDIDATES

- Write your name, centre number and candidate number in the spaces provided on the answer booklet.
- Answer **all** the questions.
- You are permitted to use a graphical calculator in this paper.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets [ ] at the end of each question or part question.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.
- Final answers should be given to a degree of accuracy appropriate to the context.
- The total number of marks for this paper is 72.

This question paper consists of 5 printed pages and 3 blank pages.

Section A (36 marks)

1 In the table below the values of x are exact and the values of the function $f(x)$ are correct to 4 decimal places.

x	2	2.1	2.2	2.3	2.4
$f(x)$	4.0000	4.2871	4.5948	4.9246	5.2780

(i) Obtain an estimate of the gradient at $x = 2$ using the forward difference formula with $h = 0.4$.

Find two further estimates by successively halving h. [4]

(ii) By considering how the differences between these three estimates reduce, obtain a better estimate, giving your answer to an appropriate degree of accuracy. [4]

2 (i) Explain by means of a simple example the difference between rounding and chopping when numbers are represented in a calculator or computer. [1]

(ii) A cheap calculator stores numbers in the range $(0, 1)$ in decimal form rounded to 7 decimal places.

The number $\frac{2}{3}$ is stored in decimal form. Write down the absolute error in the stored value.

Write down the maximum possible error in the stored value of x, where $0.1 < x < 1$. Find the maximum possible relative error in the stored value of x. [6]

3 Show that the equation

$$x^5 - 3x - 2 = 0$$

has a root, α, in the interval $1.4 < x < 1.5$.

Use the secant method with $x_0 = 1.4$ and $x_1 = 1.5$ to find x_2, x_3, x_4.

Give a value for α to the accuracy that is appropriate. [8]

4 Obtain estimates of the value of

$$I = \int_2^4 \sqrt{(1+\sqrt{x})}\,\mathrm{d}x$$

using the mid-point rule and the trapezium rule, each with $h = 2$. Give your answers to 6 decimal places.

Hence obtain a Simpson's rule estimate of I.

You are now given that the Simpson's rule estimates of I obtained as above but starting with $h = 1$ and $h = 0.5$ are 3.299 231 and 3.299 238. By considering the differences in the three Simpson's rule estimates, obtain the best estimate you can. Give your answer to an appropriate number of significant figures. [8]

5 The series

$$1 + (\tfrac{1}{2})^2 + (\tfrac{1}{3})^2 + \ldots + (\tfrac{1}{10\,000})^2$$

is being summed in a computer program that works to a precision of 7 significant figures.

When the terms are summed in the order shown, the value obtained is 1.644 725. When the same terms are summed in reverse order, the value obtained is 1.644 834.

Explain why computations of this type do not give exact answers.

Explain further why the two approaches used give different answers. State which of the answers is likely to be more accurate and explain why. [5]

[Turn over

Section B (36 marks)

6 **(i)** Copy and complete the following difference table as far as third differences.

x	$\mathrm{f}(x)$	$\Delta\mathrm{f}(x)$	$\Delta^2\mathrm{f}(x)$	$\Delta^3\mathrm{f}(x)$
1	4			
		-3		
2	1		·	
		3		·
3	4		·	
		$a-4$		·
4	a		·	
		$76-a$		
5	76			

[4]

(ii) Given that $\mathrm{f}(x)$ is a cubic polynomial, show that $a = 25$.

Use Newton's forward difference interpolation formula to obtain $\mathrm{f}(x)$. [8]

(iii) Find, correct to 3 significant figures, the minimum value of $\mathrm{f}(x)$ for $1 < x < 5$. [3]

(iv) The cubic polynomial $\mathrm{f}(x)$ could have been found using only the following data.

x	1	2	3	5
$\mathrm{f}(x)$	4	1	4	76

Write down, *but do not simplify*, an expression for $\mathrm{f}(x)$ based on these data. [3]

7 **(i)** Sketch, on the same axes, the graphs with equations

$$y = 3 - x,$$

$$y = 4\cos x,$$

where x is in radians and $0 \leqslant x \leqslant 2\pi$.

Hence show that the equation

$$3 - x = 4\cos x \qquad (*)$$

has two roots, α and β with $\alpha < \beta$, in the interval $[0, 2\pi]$. [3]

(ii) Use the iteration

$$x_{r+1} = \arccos\left(\tfrac{1}{4}(3 - x_r)\right)$$

to find the root α correct to 3 decimal places. [4]

(iii) By considering the ratio of the differences between the values of x_r, show that the iteration in part **(ii)** displays first order convergence. [3]

(iv) Given that the derivative of $\cos x$ is $(-\sin x)$ when x is in radians, obtain the following Newton-Raphson iteration for equation (*).

$$x_{r+1} = x_r + \frac{3 - x_r - 4\cos x_r}{1 - 4\sin x_r}.$$

Use this iteration to find β correct to 4 decimal places. [5]

(v) By considering the ratio of the differences between the values of x_r, show that the iteration in part **(iv)** displays convergence that is faster than first order. [3]

Mark Scheme

1(i)	h	0.4	0.2	0.1			
	est f '(2)	3.195	2.974	2.871		**M1A1A1A1**	**[4]**
(ii)	differences		-0.221	-0.103		**M1A1**	

differences approximately halving so extrapolate to 2.871 - 0.103 = 2.768.
Last figure unreliable so 2.77. Accept argument to 2.8. **M1A1** **[4]**

2(i)	E.g. 2/3 rounded to 0.666 666 7, chopped to 0.666 666 6		**E1**	**[1]**
(ii)	2/3 stored as 0.666 666 7	Absolute error 0.000 000 033...	**A1A1**	
	mpe is 0.000 000 05		**A1**	
	mpre is greatest when x is least		**M1**	
	mpre is 0.000 000 05 / 0.1 =	5 * 10^ -7 5E-07	**M1A1**	**[6]**

3	x	f(x)			
	1.4	-0.82176			
	1.5	1.09375	root in the interval (1.4, 1.5)	**B1**	
		r	Xr	f(Xr)	
		0	1.4	-0.82176	
		1	1.5	1.09375	**M1**
		2	1.4429	-0.07436	**M1A1**
		3	1.446535	-0.00609	**A1**
		4	1.446859	3.88E-05	**M1A1**

Root at 1.447 seems secure. **B1** **[8]**

4	x	f(x)				
	2	1.553774	M =		3.305783	**M1A1**
	3	1.652892	T =		3.285825	**A1**
	4	1.732051	S =	(2*M + T) / 3 =	3.299130	**M1A1**

S(h=2)	3.299130	diffs
S(h=1)	3.299231	0.0001
S(h=0.5)	3.299238	7 E -06

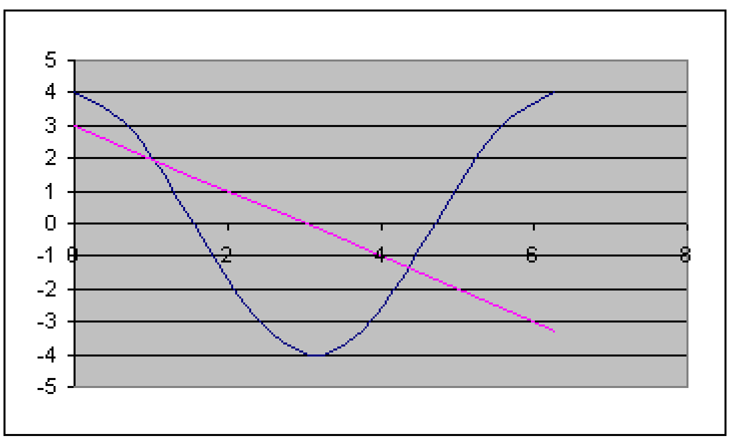
Differences reducing very rapidly. 3.29924 seems secure. **M1A1A1** **[8]**

5	Computations of this type contain rounding errors	**E1**
	The rounding errors will be different when the two sums are computed	**E1**
	Adding from large to small loses precision (the small number is lost)	**E1**
	Adding from small to large allows each number to contribute to the sum	**E1**
	Hence the second sum is likely to be more accurate	**E1**

	x	f(x)	Δf(x)	$\Delta^2 f(x)$	$\Delta^3 f(x)$		
6(i)	1	4					
			–3				
	2	1		**6**			
			3		**a – 13**		
	3	4		**a – 7**			
			a – 4		**87 – 3a**		
	4	a		**80 – 2a**		**A4**	
			76 – a			**(-1 each**	
	5	76				**error)**	**[4]**

(ii)	$87 - 3a = a - 13$ gives $a = 25$	**M1**	
	$f(x) = 4 - 3(x-1) + 6(x-1)(x-2)/2 + 12(x-1)(x-2)(x-3)/6$	**M1A1A1A1A1**	
	$= 4 - 3x + 3 + 3x^2 - 9x + 6 + 2x^3 - 12x^2 + 22x - 12$	**A1**	
	$= 2x^3 - 9x^2 + 10x + 1$	**A1**	**[8]**
(iii)	$f'(x) = 6x^2 - 18x + 10 = 0$ — *Algebra may appear in (iii) rather than (ii) for full credit*	**M1**	
	$x = 2.26 \quad (2.26376)$	**A1**	
	$f(2.26...) = 0.718$	**A1**	**[3]**
(iv)	$f(x) = 4(x - 2)(x - 3)(x - 5)/(1 - 2)(1 - 3)(1 - 5)$ + three similar terms	**M1A1A1**	**[3]**
			Total 18

7(i)



	G1	
	G1	
shows two roots	**E1**	**[3]**

(ii)	E.g.:								
	r	0	1	2	3	4	5	6	7
	Xr	1	1.04720	1.06077	1.06465	1.06576	1.06608	1.06617	1.06620
								M1A1A1	
	alpha = 1.066 correct to 3 decimal places							**A1**	**[4]**
(iii)	0	1	2	3	4	5	6		
	1	1.04720	1.06077	1.06465	1.06576	1.06608	1.06617		
	diffs	0.04720	0.01357	0.00388	0.00111	0.00032	0.00009		
	ratio of diffs		0.28756	0.28615	0.28575	0.28564	0.28561	**M1A1**	
	ratios (approx) constant so first order convergence.							**E1**	**[3]**
(iv)	Obtain N-R iteration (beware printed answer)							**M1A1**	
	E.g.:								
	r	0	1	2	3	4			
	Xr	5	4.35177	4.36435	4.36432	4.36432		**M1A1**	
	beta = 4.3643 correct to 4 decimal places							**A1**	**[5]**
(v)	0	1	2	3	4				
	5	4.35177	4.36435	4.36432	4.36432				
	diffs	-0.64823	0.01258	-0.00002	0.00000				
	ratio of diffs		-0.01940	-0.00184	0.00000			**M1A1**	
	ratios getting (much) smaller so faster than first order							**E1**	**[3]**
									Total 18