## ADVANCED GCE <br> MATHEMATICS (MEI)

Mechanics 3

Candidates answer on the answer booklet.
OCR supplied materials:

- 8 page answer booklet (sent with general stationery)
- MEI Examination Formulae and Tables (MF2)

Other materials required:

- Scientific or graphical calculator

Wednesday 22 June 2011
Morning
Duration: 1 hour 30 minutes


## INSTRUCTIONS TO CANDIDATES

- Write your name, centre number and candidate number in the spaces provided on the answer booklet. Please write clearly and in capital letters.
- Use black ink. Pencil may be used for graphs and diagrams only.
- Read each question carefully. Make sure you know what you have to do before starting your answer.
- Answer all the questions.
- Do not write in the bar codes.
- You are permitted to use a scientific or graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.
- The acceleration due to gravity is denoted by $g \mathrm{~m} \mathrm{~s}^{-2}$. Unless otherwise instructed, when a numerical value is needed, use $g=9.8$.


## INFORMATION FOR CANDIDATES

- The number of marks is given in brackets [ ] at the end of each question or part question.
- You are advised that an answer may receive no marks unless you show sufficient detail of the working to indicate that a correct method is being used.
- The total number of marks for this paper is 72.
- This document consists of $\mathbf{8}$ pages. Any blank pages are indicated.

1 A particle is moving in a straight line. At time $t$ its displacement $x$ from a fixed point O on the line is given by

$$
x=A \sin \omega t
$$

where $A$ and $\omega$ are constants.
(i) Show that $\frac{\mathrm{d}^{2} x}{\mathrm{~d} t^{2}}=-\omega^{2} x$ and $\left(\frac{\mathrm{d} x}{\mathrm{~d} t}\right)^{2}=\omega^{2}\left(A^{2}-x^{2}\right)$.

A ball floats on the surface of the sea. Waves cause the ball to rise and fall in a vertical line, and the ball is executing simple harmonic motion. The centre of the oscillations is 8 m above the sea-bed. The ball has speed $1.2 \mathrm{~m} \mathrm{~s}^{-1}$ when it is 7.3 m above the sea-bed, and it has speed $0.75 \mathrm{~m} \mathrm{~s}^{-1}$ when it is 10 m above the sea-bed.
(ii) Show that the amplitude of the oscillations is 2.5 m , and find the period.
(iii) Find the maximum speed of the ball.
(iv) Find the magnitude and direction of the acceleration of the ball when it is 6.4 m above the sea-bed.
(v) Find the time taken for the ball to move upwards from 6 m above the sea-bed to 9 m above the sea-bed.

2 (a) A particle P of mass 0.2 kg is connected to a fixed point O by a light inextensible string of length 3.2 m , and is moving in a vertical circle with centre O and radius 3.2 m . Air resistance may be neglected. When P is at the highest point of the circle, the tension in the string is 0.6 N .
(i) Find the speed of P when it is at the highest point.
(ii) For an instant when OP makes an angle of $60^{\circ}$ with the downward vertical, find
(A) the radial and tangential components of the acceleration of P ,
$(B)$ the tension in the string.
(b) A solid cone is fixed with its axis of symmetry vertical and its vertex V uppermost. The semivertical angle of the cone is $36^{\circ}$, and its surface is smooth. A particle $Q$ of mass 0.2 kg is connected to V by a light inextensible string, and Q moves in a horizontal circle at constant speed, in contact with the surface of the cone, as shown in Fig. 2.


Fig. 2

The particle Q makes one complete revolution in 1.8 s , and the normal reaction of the cone on Q has magnitude 0.75 N .
(i) Find the tension in the string.
(ii) Find the length of the string.

3 Fixed points A and B are 4.8 m apart on the same horizontal level. The midpoint of AB is M . A light elastic string, with natural length 3.9 m and modulus of elasticity 573.3 N , has one end attached to A and the other end attached to $B$.
(i) Find the elastic energy stored in the string.

A particle P is attached to the midpoint of the string, and is released from rest at M . It comes instantaneously to rest when P is 1.8 m vertically below M .
(ii) Show that the mass of P is 15 kg .
(iii) Verify that P can rest in equilibrium when it is 1.0 m vertically below M .

In general, a light elastic string, with natural length $a$ and modulus of elasticity $\lambda$, has its ends attached to fixed points which are a distance $d$ apart on the same horizontal level. A particle of mass $m$ is attached to the midpoint of the string, and in the equilibrium position each half of the string has length $h$, as shown in Fig. 3.


Fig. 3

When the particle makes small oscillations in a vertical line, the period of oscillation is given by the formula

$$
\sqrt{\frac{8 \pi^{2} h^{3}}{8 h^{3}-a d^{2}}} m^{\alpha} a^{\beta} \lambda^{\gamma} .
$$

(iv) Show that $\frac{8 \pi^{2} h^{3}}{8 h^{3}-a d^{2}}$ is dimensionless.
(v) Use dimensional analysis to find $\alpha, \beta$ and $\gamma$.
(vi) Hence find the period when the particle P makes small oscillations in a vertical line centred on the position of equilibrium given in part (iii).

4 The region $A$ is bounded by the curve $y=x^{2}+5$ for $0 \leqslant x \leqslant 3$, the $x$-axis, the $y$-axis and the line $x=3$. The region $B$ is bounded by the curve $y=x^{2}+5$ for $0 \leqslant x \leqslant 3$, the $y$-axis and the line $y=14$. These regions are shown in Fig. 4.


Fig. 4
(i) Find the coordinates of the centre of mass of a uniform lamina occupying the region $A$.
(ii) The region $B$ is rotated through $2 \pi$ radians about the $y$-axis to form a uniform solid of revolution $R$. Find the $y$-coordinate of the centre of mass of the solid $R$.
(iii) The region $A$ is rotated through $2 \pi$ radians about the $y$-axis to form a uniform solid of revolution $S$. Using your answer to part (ii), or otherwise, find the $y$-coordinate of the centre of mass of the solid $S$.

| 1 (i) | $\begin{aligned} \frac{\mathrm{d} x}{\mathrm{~d} t} & =A \omega \cos \omega t \\ \frac{\mathrm{~d}^{2} x}{\mathrm{~d} t^{2}} & =-A \omega^{2} \sin \omega t \\ & =-\omega^{2}(A \sin \omega t)=-\omega^{2} x \\ \left(\frac{\mathrm{~d} x}{\mathrm{~d} t}\right)^{2} & =A^{2} \omega^{2} \cos ^{2} \omega t=A^{2} \omega^{2}\left(1-\sin ^{2} \omega t\right) \\ & =\omega^{2}\left(A^{2}-A^{2} \sin ^{2} \omega t\right)=\omega^{2}\left(A^{2}-x^{2}\right) \end{aligned}$ | B1 <br> M1 <br> E1 <br> M1 <br> E1 <br> 5 | Obtaining second derivative <br> Using $\cos ^{2} \omega t=1-\sin ^{2} \omega t$ |
| :---: | :---: | :---: | :---: |
| (ii) | $\begin{aligned} 1.2^{2} & =\omega^{2}\left(A^{2}-0.7^{2}\right) \\ 0.75^{2} & =\omega^{2}\left(A^{2}-2^{2}\right) \end{aligned}$ $\begin{aligned} \frac{A^{2}-0.49}{A^{2}-4} & =\frac{1.2^{2}}{0.75^{2}} \\ A^{2}-0.49 & =2.56\left(A^{2}-4\right) \\ 9.75 & =1.56 A^{2} \\ A^{2} & =6.25 \end{aligned}$ <br> Amplitude is 2.5 m $\begin{aligned} 1.44 & =\omega^{2}\left(2.5^{2}-0.7^{2}\right) \\ \omega & =0.5 \end{aligned}$ <br> Period is $\frac{2 \pi}{\omega}=\frac{2 \pi}{0.5}$ $\begin{equation*} =4 \pi=12.6 \mathrm{~s} \tag{3sf} \end{equation*}$ | M1 <br> A1 <br> M1 <br> E1 <br> M1 <br> A1 | Using $v^{2}=\omega^{2}\left(A^{2}-x^{2}\right)$ <br> Two correct equations <br> (M0 if $x=7.3$ used, etc) <br> Eliminating $\omega$ <br> or Eliminating $A$ <br> or Substituting $A=2.5$ into both equations <br> Correctly shown <br> Using $\frac{2 \pi}{\omega}$ |
| (iii) | Maximum speed is $A \omega=1.25 \mathrm{~ms}^{-1}$ | B1 $1$ | ft only if greater than 1.2 |
| (iv) | Magnitude is $0.5^{2} \times 1.6$ $=0.4 \mathrm{~m} \mathrm{~s}^{-2}$ <br> Direction is upwards | M1 <br> A1 B1 <br> 3 | Accept -0.4 <br> B0 for just 'towards centre' |
| (v) | $x=2.5 \sin (0.5 t)$ <br> When $x=-2, \quad t=-1.855$ (or 10.71) <br> When $x=1, t=0.823$ (or 13.39) <br> Time taken is $0.823-(-1.855)$ $=2.68 \mathrm{~s} \quad(3 \mathrm{sf})$ | B1 <br> M1 <br> A1 <br> 3 | or $x=2.5 \cos (0.5 t)$ <br> or $t=( \pm) 4.996$ <br> or $t=( \pm) 2.319$ <br> Correct strategy for finding time (must use radians) <br> (ft is $1.3388 / \omega$ ) |


| 2(a)(i) | $0.6+0.2 \times 9.8=0.2 \times \frac{u^{2}}{3.2}$ <br> Speed is $6.4 \mathrm{~ms}^{-1}$ | M1 <br> A1 <br> A1 <br> 3 | For acceleration $\frac{u^{2}}{3.2}$ |
| :---: | :---: | :---: | :---: |
| (ii) | (A) $\begin{aligned} \frac{1}{2} m\left(v^{2}-u^{2}\right) & =m \times 9.8\left(3.2+3.2 \cos 60^{\circ}\right) \\ v^{2} & =135.04 \end{aligned}$ <br> Radial component is $\frac{v^{2}}{3.2}=42.2 \mathrm{~m} \mathrm{~s}^{-2}$ <br> Tangential component is $g \sin 60^{\circ}$ $\begin{equation*} =8.49 \mathrm{~m} \mathrm{~s}^{-2} \tag{3sf} \end{equation*}$ | M1 <br> A1 <br> A1 <br> M1 <br> A1 $5$ | Equation involving KE and PE <br> ( ft is $29.4+\frac{u^{2}}{3.2}$ ) <br> M1A0 for $g \cos 60^{\circ}$ <br> M0 for $m g \sin 60^{\circ}$ <br> If radial and tangential components are interchanged, withhold first A1 |
|  | (B) $T-m g \cos 60^{\circ}=m a$ $T-0.2 \times 9.8 \cos 60^{\circ}=0.2 \times 42.2$ <br> Tension is 9.42 N | M1 <br> A1 <br> A1 cao | Radial equation (three terms) (Allow M1 for $T-m g=m a$ ) This M1 can be awarded in (A) ft dependent on M1 for energy in (A) SC If $60^{\circ}$ with upward vertical, <br> (A) M1A0A0 M1A1 (8.49) <br> (B) M1A1A1 (3.54) |
| (b)(i) | $T \cos 36^{\circ}+0.75 \sin 36^{\circ}=0.2 \times 9.8$ <br> Tension is $1.88 \mathrm{~N} \quad(3 \mathrm{sf})$ | M1 <br> A1 <br> 2 | Resolving vertically (three terms) Allow sin/cos confusion, but both $T$ and $R$ must be resolved |
| (ii) | Angular speed $\omega=\frac{2 \pi}{1.8} \quad(=3.491)$ $\begin{aligned} T \sin 36^{\circ}-0.75 \cos 36^{\circ} & =0.2 r\left(\frac{2 \pi}{1.8}\right)^{2} \\ r & =0.204 \end{aligned}$ <br> Length of string is $\frac{r}{\sin 36^{\circ}}$ $\begin{equation*} =0.347 \mathrm{~m} \tag{3sf} \end{equation*}$ | B1 <br> M1 <br> A1 <br> M1 <br> A1 cao | Or $v=\frac{2 \pi r}{1.8}$ <br> Horiz eqn involving $r \omega^{2}$ or $v^{2} / r$ <br> Equation for $r$ (or l) <br> Dependent on previous M1 |


| 3 (i) | $\text { Elastic energy is } \begin{gathered} \frac{1}{2} \times \frac{573.3}{3.9} \times 0.9^{2} \\ =59.535 \mathrm{~J} \end{gathered}$ | $\begin{aligned} & \mathrm{M} 1 \\ & \mathrm{~A} 1 \end{aligned}$ $2$ | Allow one error <br> (Allow 60 A0 for 59) |
| :---: | :---: | :---: | :---: |
| (ii) | Length of string at bottom is $2 \sqrt{1.8^{2}+2.4^{2}} \quad(=6)$ $\begin{aligned} \frac{1}{2} \times \frac{573.3}{3.9} \times\left(2.1^{2}-0.9^{2}\right) & =m \times 9.8 \times 1.8 \\ 324.135-59.535 & =17.64 m \end{aligned}$ <br> Mass is 15 kg | M1 <br> M1 <br> B1B1 <br> E1 <br> 5 | Finding length of string (or half-string) <br> Equation involving EE and PE For change in EE and change in PE |
| (iii) | Length of string is $2 \sqrt{1.0^{2}+2.4^{2}}=5.2$ <br> Tension $T=\frac{573.3}{3.9} \times 1.3 \quad(=191.1)$ $\begin{aligned} 2 T \sin \alpha-m g & =2 \times 191.1 \times \frac{1.0}{2.6}-15 \times 9.8 \\ & =147-147 \\ & =0, \text { hence it is in equilibrium } \end{aligned}$ | M1 <br> A1 <br> M1 <br> E1 | Finding tension (via Hooke's law) <br> Finding vertical component of tension Give A1 for $T=191.1$ obtained from resolving vertically <br> SC If 573.3 is used as stiffness: <br> (i) M1A0 (ii) M1M1B0B1E0 <br> (iii) M1A1 (745.29) M1E0 |
| (iv) | $\left[8 \pi^{2} h^{3}\right]=\mathrm{L}^{3}, \quad\left[8 h^{3}-a d^{2}\right]=\mathrm{L}^{3}$ <br> So $\frac{8 \pi^{2} h^{3}}{8 h^{3}-a d^{2}}$ is dimensionless | E1 | Condone ' $\mathrm{L}^{3} / \mathrm{L}^{3}=0$, dimensionless' But EO for $\frac{L^{3}}{\mathrm{~L}^{3}-\mathrm{L}^{3}}=\frac{\mathrm{L}^{3}}{0}$ |
| (v) | $\begin{aligned} & \mathrm{T}=\mathrm{M}^{\alpha} \mathrm{L}^{\beta}\left(\mathrm{MLT}^{-2}\right)^{\gamma} \\ & \gamma=-\frac{1}{2} \\ & \alpha+\gamma=0, \text { so } \alpha=\frac{1}{2} \\ & \beta+\gamma=0, \text { so } \beta=\frac{1}{2} \end{aligned}$ | B1 <br> B1 <br> B1 <br> B1 <br> 4 | For $[\lambda]=\mathrm{MLT}^{-2}$ <br> If $\gamma$ is wrong but non-zero, give B 1 ft for $\alpha=\beta=-\gamma$ |
| (vi) | $a=3.9, \lambda=573.3, d=4.8, h=2.6, m=15$ <br> Period is $\sqrt{\frac{8 \pi^{2} h^{3}}{8 h^{3}-a d^{2}}} m^{1 / 2} a^{1 / 2} \lambda^{-1 / 2}=1.67 \mathrm{~s}$ | M1 <br> A1 cao 2 | Using formula with numerical $\alpha, \beta, \gamma$ (must use the complete formula) |


| 4 (i) |  | M1 <br> A1 <br> M1 <br> A1 <br> A1 <br> M1 <br> A2 <br> A1 | For $\int\left(x^{2}+5\right) \mathrm{d} x$ <br> For $\frac{1}{3} x^{3}+5 x$ <br> For $\int x y \mathrm{~d} x$ <br> For $\frac{1}{4} x^{4}+\frac{5}{2} x^{2}$ <br> For $\int y^{2} \mathrm{~d} x$ <br> For $\frac{1}{10} x^{5}+\frac{5}{3} x^{3}+\frac{25}{2} x$ <br> Give A1 for two terms correct |
| :---: | :---: | :---: | :---: |
| (ii) |  | M1 <br> A1 <br> M1 <br> A1 <br> M1 <br> A1 | For $\int(y-5) \mathrm{d} y$ <br> For $\left[\frac{1}{2} y^{2}-5 y\right]_{5}^{14}$ <br> For $\int x^{2} y \mathrm{~d} x$ <br> For $\frac{1}{3} y^{3}-\frac{5}{2} y^{2}$ <br> Dependent on previous M1M1 |
| (iii) | Volume of whole cylinder is $\pi \times 3^{2} \times 14=126 \pi$ $\begin{align*} 126 \pi \times 7 & =40.5 \pi \times 11+(126 \pi-40.5 \pi) \times \bar{y}_{A} \\ \bar{y}_{A} & =\frac{126 \pi \times 7-40.5 \pi \times 11}{126 \pi-40.5 \pi} \\ & =\frac{97}{19}=5.105 \quad(4 \mathrm{sf}) \tag{4sf} \end{align*}$ | M1 <br> A1 <br> A1 cao $3$ | Using formula for composite body |

