

**ADVANCED GCE UNIT  
MATHEMATICS (MEI)**

Mechanics 3

**WEDNESDAY 10 JANUARY 2007**

**4763/01**

Afternoon  
Time: 1 hour 30 minutes

Additional materials:  
Answer booklet (8 pages)  
Graph paper  
MEI Examination Formulae and Tables (MF2)

**INSTRUCTIONS TO CANDIDATES**

- Write your name, centre number and candidate number in the spaces provided on the answer booklet.
- Answer **all** the questions.
- You are permitted to use a graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.
- The acceleration due to gravity is denoted by  $g \text{ m s}^{-2}$ . Unless otherwise instructed, when a numerical value is needed, use  $g = 9.8$ .

**INFORMATION FOR CANDIDATES**

- The number of marks is given in brackets [ ] at the end of each question or part question.
- The total number of marks for this paper is 72.

**ADVICE TO CANDIDATES**

- Read each question carefully and make sure you know what you have to do before starting your answer.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.

- 1 (i) Write down the dimensions of velocity, acceleration and force. [3]

The force  $F$  of gravitational attraction between two objects with masses  $m_1$  and  $m_2$ , at a distance  $r$  apart, is given by

$$F = \frac{Gm_1m_2}{r^2}$$

where  $G$  is the universal constant of gravitation.

- (ii) Show that the dimensions of  $G$  are  $\text{M}^{-1}\text{L}^3\text{T}^{-2}$ . [2]

- (iii) In SI units (based on the kilogram, metre and second) the value of  $G$  is  $6.67 \times 10^{-11}$ .

Find the value of  $G$  in imperial units based on the pound (0.4536 kg), foot (0.3048 m) and second. [3]

- (iv) For a planet of mass  $m$  and radius  $r$ , the escape velocity  $v$  from the planet's surface is given by

$$v = \sqrt{\frac{2Gm}{r}}.$$

Show that this formula is dimensionally consistent. [3]

- (v) For a planet in circular orbit of radius  $R$  round a star of mass  $M$ , the time  $t$  taken to complete one orbit is given by

$$t = kG^\alpha M^\beta R^\gamma$$

where  $k$  is a dimensionless constant.

Use dimensional analysis to find  $\alpha$ ,  $\beta$  and  $\gamma$ . [5]

- 2 (a) A light inextensible string has length 1.8 m. One end of the string is attached to a fixed point O, and the other end is attached to a particle of mass 5 kg. The particle moves in a complete vertical circle with centre O, so that the string remains taut throughout the motion. Air resistance may be neglected.

(i) Show that, at the highest point of the circle, the speed of the particle is at least  $4.2 \text{ m s}^{-1}$ . [3]

(ii) Find the least possible tension in the string when the particle is at the lowest point of the circle. [5]

- (b) Fig. 2 shows a hollow cone mounted with its axis of symmetry vertical and its vertex V pointing downwards. The cone rotates about its axis with a constant angular speed of  $\omega \text{ rad s}^{-1}$ . A particle P of mass 0.02 kg is in contact with the rough inside surface of the cone, and does not slip. The particle P moves in a horizontal circle of radius 0.32 m. The angle between VP and the vertical is  $\theta$ .

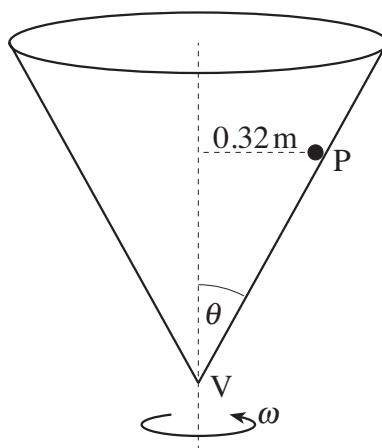


Fig. 2

In the case when  $\omega = 8.75$ , there is no frictional force acting on P.

(i) Show that  $\tan \theta = 0.4$ . [4]

Now consider the case when  $\omega$  takes a constant value greater than 8.75.

(ii) Draw a diagram showing the forces acting on P. [2]

(iii) You are given that the coefficient of friction between P and the surface is 0.11. Find the maximum possible value of  $\omega$  for which the particle does not slip. [6]

- 3 Ben has mass 60 kg and he is considering doing a bungee jump using an elastic rope with natural length 32 m. One end of the rope is attached to a fixed point O, and the other end is attached to Ben. When Ben is supported in equilibrium by the rope, the length of the rope is 32.8 m.

To predict what will happen, Ben is modelled as a particle B, the rope is assumed to be light, and air resistance is neglected. B is released from rest at O and falls vertically. When the rope becomes stretched,  $x$  m denotes the extension of the rope.

(i) Find the stiffness of the rope. [2]

(ii) Use an energy argument to show that, when B comes to rest instantaneously with the rope stretched,

$$x^2 - 1.6x - 51.2 = 0.$$

Hence find the length of the rope when B is at its lowest point. [6]

(iii) Show that, while the rope is stretched,

$$\frac{d^2x}{dt^2} + 12.25x = 9.8$$

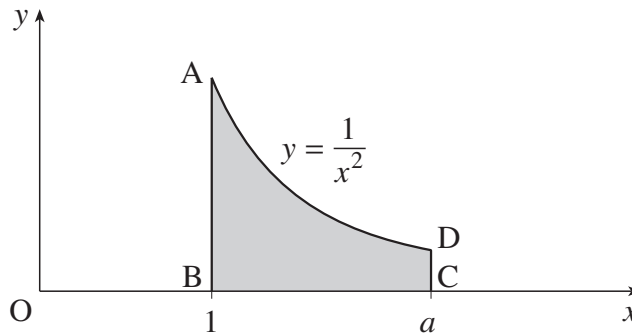
where  $t$  is the time measured in seconds. [4]

(iv) Find the time taken for B to travel between the equilibrium position ( $x = 0.8$ ) and the lowest point. [3]

(v) Find the acceleration of B when it is at the lowest point, and comment on the implications for Ben. [3]

- 4 In this question,  $a$  is a constant with  $a > 1$ .

Fig. 4 shows the region bounded by the curve  $y = \frac{1}{x^2}$  for  $1 \leq x \leq a$ , the  $x$ -axis, and the lines  $x = 1$  and  $x = a$ .



**Fig. 4**

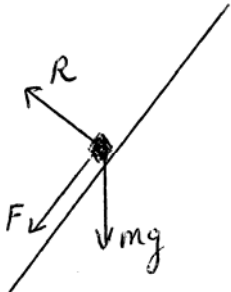
This region is occupied by a uniform lamina ABCD, where A is  $(1, 1)$ , B is  $(1, 0)$ , C is  $(a, 0)$  and D is  $\left(a, \frac{1}{a^2}\right)$ . The centre of mass of this lamina is  $(\bar{x}, \bar{y})$ .

- (i) Find  $\bar{x}$  in terms of  $a$ , and show that  $\bar{y} = \frac{a^3 - 1}{6(a^3 - a^2)}$ . [8]
- (ii) In the case  $a = 2$ , the lamina is freely suspended from the point A, and hangs in equilibrium. Find the angle which AB makes with the vertical. [3]

The region shown in Fig. 4 is now rotated through  $2\pi$  radians about the  $x$ -axis to form a uniform solid of revolution.

- (iii) Find the  $x$ -coordinate of the centre of mass of this solid of revolution, in terms of  $a$ , and show that it is less than 1.5. [7]

<b>1 (i)</b>	$[ \text{Velocity} ] = \text{L T}^{-1}$ $[ \text{Acceleration} ] = \text{L T}^{-2}$ $[ \text{Force} ] = \text{M L T}^{-2}$	B1 B1 B1 <b>3</b>	<i>Deduct 1 mark if answers given as</i> $\text{ms}^{-1}, \text{ms}^{-2}, \text{kg ms}^{-2}$
<b>(ii)</b>	$[ G ] = \frac{[ F ][ r^2 ]}{[ m_1 ][ m_2 ]} = \frac{(\text{M L T}^{-2})(\text{L}^2)}{\text{M}^2}$ $= \text{M}^{-1} \text{L}^3 \text{T}^{-2}$	M1 E1 <b>2</b>	
<b>(iii)</b>	$G = 6.67 \times 10^{-11} \times 0.4536 \times \frac{1}{(0.3048)^3}$ $= 1.07 \times 10^{-9} \quad (\text{lb}^{-1} \text{ft}^3 \text{s}^{-2})$	M1M1 A1 <b>3</b>	For $\times 0.4536$ and $\times \frac{1}{(0.3048)^3}$ SC Give M1 for $6.67 \times 10^{-11} \times \frac{1}{0.4536} \times (0.3048)^3$ $(= 4.16 \times 10^{-12})$
<b>(iv)</b>	$[ \text{RHS} ] = \sqrt{\frac{(\text{M}^{-1} \text{L}^3 \text{T}^{-2})(\text{M})}{\text{L}}}$ $= \sqrt{\text{L}^2 \text{T}^{-2}} = \text{L T}^{-1}$ which is the same as [ LHS ]	M1A1 E1 <b>3</b>	
<b>(v)</b>	$T = (\text{M}^{-1} \text{L}^3 \text{T}^{-2})^\alpha \text{M}^\beta \text{L}^\gamma$ Powers of M: $-\alpha + \beta = 0$ of L: $3\alpha + \gamma = 0$ of T: $-2\alpha = 1$ $\alpha = -\frac{1}{2}, \quad \beta = -\frac{1}{2}, \quad \gamma = \frac{3}{2}$	M1 M1 A1 M1 A1 <b>5</b>	At least two equations Three correct equations Obtaining at least one of $\alpha, \beta, \gamma$

2(a)(i)	<p>At the highest point,</p> $T + 5 \times 9.8 = 5 \times \frac{v^2}{1.8}$ <p>For least speed, <math>T = 0</math>, <math>v^2 = 1.8 \times 9.8</math>  Speed is at least <math>4.2 \text{ ms}^{-1}</math></p>	<p>M1 A1 E1 <b>3</b></p>	<p>Using acceleration <math>v^2 / 1.8</math>   <i>T</i> may be omitted</p>
(ii)	<p>For least tension, speed at top is <math>4.2 \text{ ms}^{-1}</math>  By conservation of energy,  <math display="block">\frac{1}{2} \times 5 \times (w^2 - 4.2^2) = 5 \times 9.8 \times 3.6</math> <math display="block">w^2 = 88.2 \quad (w = 9.39)</math> <math display="block">T - 5 \times 9.8 = 5 \times \frac{88.2}{1.8}</math> Tension is at least 294 N</p>	<p>M1 A1  M1 A1 ft A1 <b>5</b></p>	<p>Energy equation with 3 terms   Equation of motion with 3 terms</p>
(b)(i)	$R \sin \theta = 0.02 \times 9.8$ $R \cos \theta = 0.02 \times 0.32 \times 8.75^2$ $\tan \theta = \frac{0.02 \times 9.8}{0.02 \times 0.32 \times 8.75^2} = 0.4$	<p>B1 M1 A1 E1 <b>4</b></p>	<p>Using acceleration <math>0.32 \times 8.75^2</math>  SC If <math>\sin \theta</math> and <math>\cos \theta</math> interchanged, award B0M1A1E0</p>
(ii)		<p>B1 B1 <b>2</b></p>	<p>For <math>R</math> and <math>mg</math>  For <math>F</math> acting down the slope</p>
(iii)	$R \sin \theta = 0.02 \times 9.8 + F \cos \theta$ $R \cos \theta + F \sin \theta = 0.02 \times 0.32 \omega^2$ <p>For maximum <math>\omega</math>, <math>F = \mu R</math>  <math display="block">R(\sin \theta - \mu \cos \theta) = 0.02 \times 9.8</math> <math display="block">R(\cos \theta + \mu \sin \theta) = 0.02 \times 0.32 \omega^2</math> <math display="block">\omega^2 = \frac{9.8(\cos \theta + \mu \sin \theta)}{0.32(\sin \theta - \mu \cos \theta)} = \frac{9.8(1 + \mu \tan \theta)}{0.32(\tan \theta - \mu)}</math> <math display="block">= \frac{9.8(1 + 0.11 \times 0.4)}{0.32(0.4 - 0.11)}</math> <math display="block">\omega = 10.5</math> </p>	<p>M1 A1 A1 M1    M1 A1 cao <b>6</b></p>	<p>Resolving <math>F</math> and <math>R</math> [ or <math>mg</math> and accn ]  <i>Can give A1A1 for sin / cos interchanged consistent with (i)</i>  <i>Dependent on first M1</i>   Obtaining a numerical value for <math>\omega^2</math>  <i>Dependent on M1M1</i></p>

<b>3 (i)</b>	$k \times 0.8 = 60 \times 9.8$ Stiffness is $735 \text{ N m}^{-1}$	M1 A1 <b>2</b>	
<b>(ii)</b>	Loss of PE is $60 \times 9.8(32 + x)$ Gain in EE is $\frac{1}{2} \times 735x^2$ $\frac{1}{2} \times 735x^2 = 60 \times 9.8(32 + x)$ $x^2 = 1.6(32 + x)$ $x^2 - 1.6x - 51.2 = 0$ $(x - 8)(x + 6.4) = 0$ $x = 8$ Length of rope is $40 \text{ m}$	B1 B1 M1  E1  M1 A1 <b>6</b>	<i>If x is measured from equilibrium position, treat as MR</i>  Obtaining a value of x
<b>(iii)</b>	Tension $T = 735x$ $mg - T = m \frac{d^2x}{dt^2}$ $60 \times 9.8 - 735x = 60 \frac{d^2x}{dt^2}$ $\frac{d^2x}{dt^2} + 12.25x = 9.8$	B1  M1  A1  E1 <b>4</b>	Equation of motion with 3 terms
<b>(iv)</b>	SHM with $\omega^2 = 12.25$ ( $\omega = 3.5$ ) Time taken is $\frac{1}{4} \times \frac{2\pi}{\omega}$ $= \frac{1}{7} \pi = 0.449 \text{ s}$	M1  M1  A1 <b>3</b>	or $\omega t = \frac{1}{2} \pi$
<b>(v)</b>	When $x = 8$ , $\frac{d^2x}{dt^2} = 9.8 - 12.25 \times 8$ $= -88.2$ Acceleration is $88.2 \text{ ms}^{-2}$ (upwards) This acceleration (9g) is too large for comfort	M1  A1 B1 <b>3</b>	or $735 \times 8 - 60 \times 9.8 = 60a$



<p><b>4 (i)</b></p>	<p>Area is <math>\int_1^a \frac{1}{x^2} dx = \left[ -\frac{1}{x} \right]_1^a</math></p> $= 1 - \frac{1}{a}$ <p><math>\int x y dx = \int_1^a \frac{1}{x} dx \quad (= \ln a)</math></p> $\bar{x} = \frac{\int x y dx}{\int y dx}$ $= \frac{\ln a}{1 - \frac{1}{a}} \quad \left( = \frac{a \ln a}{a - 1} \right)$ <p><math>\int \frac{1}{2} y^2 dx = \int_1^a \frac{1}{2x^4} dx = \left[ -\frac{1}{6x^3} \right]_1^a</math></p> $= \frac{1}{6} \left( 1 - \frac{1}{a^3} \right)$ <p><math>\bar{y} = \frac{\int \frac{1}{2} y^2 dx}{\int y dx}</math></p> $= \frac{\frac{1}{6} \left( 1 - \frac{1}{a^3} \right)}{1 - \frac{1}{a}} = \frac{a^3 - 1}{6(a^3 - a^2)}$	<p>M1</p> <p>A1</p> <p>M1</p> <p>M1</p> <p>A1</p> <p>M1</p> <p>M1</p> <p>E1</p>	<p>Condone omission of <math>\frac{1}{2}</math></p> <p>( <math>\frac{1}{2}</math> needed for this mark )</p> <p><b>8</b></p>
<p><b>(ii)</b></p>	<p>When <math>a = 2</math>, <math>\bar{x} = 2 \ln 2</math>, <math>\bar{y} = \frac{7}{24}</math></p> $\tan \theta = \frac{\bar{x} - 1}{1 - \bar{y}}$ $= \frac{2 \ln 2 - 1}{1 - \frac{7}{24}}$ <p><math>\theta = 28.6^\circ</math></p>	<p>M1</p> <p>A1</p> <p>A1</p>	<p>CM vertically below A</p> <p>Correct expression for <math>\tan \theta</math> or <math>\tan(90 - \theta)</math></p> <p><b>3</b></p>

(iii)	<p>Volume is <math>\int \pi y^2 dx = \pi \int_1^a \frac{1}{x^4} dx</math></p> $= \pi \left[ -\frac{1}{3x^3} \right]_1^a = \frac{\pi}{3} \left( 1 - \frac{1}{a^3} \right)$ <p><math>\int \pi x y^2 dx = \pi \int_1^a \frac{1}{x^3} dx = \pi \left[ -\frac{1}{2x^2} \right]_1^a</math></p> $= \frac{\pi}{2} \left( 1 - \frac{1}{a^2} \right)$ <p><math>\bar{x} = \frac{\int \pi x y^2 dx}{\int \pi y^2 dx}</math></p> $= \frac{\frac{\pi}{2} \left( 1 - \frac{1}{a^2} \right)}{\frac{\pi}{3} \left( 1 - \frac{1}{a^3} \right)} = \frac{3(a^3 - a)}{2(a^3 - 1)}$ <p>Since <math>a &gt; 1</math>, <math>a^3 - a &lt; a^3 - 1</math></p> <p>Hence <math>\bar{x} &lt; \frac{3}{2}</math>, i.e. <math>\bar{x} &lt; 1.5</math></p>	<p>M1</p> <p>A1</p> <p>M1</p> <p>M1</p> <p>A1</p> <p>M1</p> <p>E1</p> <p><b>7</b></p>	<p><math>\pi</math> may be omitted throughout</p> <p>Any correct form</p> <p>or <math>\bar{x} \rightarrow 1.5</math> as <math>a \rightarrow \infty</math></p> <p>Fully convincing argument</p>
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